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STUDIES ON RAINFALL-RUNOFF MODELING

6. A Statistical Analysis of Rainfall-Runoff Relationship

Partial Technical Completion Report
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6. A Statistical Analysis of Rainfall-Runoff Relationship

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PARTICAL TECHNICAL COMPLETION REPORT

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ABSTRACT

Relationships are developed between statistical parameters of annual rainfall and those of annual runoff. The transformation of rainfall into runoff is characterized by (a) a single linear reservoir, (b) a single non-linear reservoir. These relationships are compared with those available in the literature and can be used to describe the statistical distribution of runoff, given the distribution of rainfall. It is shown that, for larger carryover, the distribution of runoff always tends to be normal regardless of the distribution of rainfall and the nature of transformation process.

ACKNOWLEDGEMENT

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Chapter 1

INTRODUCTION

1.1 GENERAL REMARKS

The relationship between rainfall and runoff is one of the most important problems in hydrology. It is also one of the most difficult problems. The rainfall-runoff relationship quantifies the response function describing the behavior of a watershed. The response function is a result of numerous processes, complex and interdependent, that participate in the transformation of rainfall into runoff. The transformation process encompasses virtually the entire domain of the hydrologic cycle. This all-encompassing nature of the transformation process is largely responsible for the complexity underlying the rainfall-runoff relationship. The complexity is compounded further by spatial and temporal variability of hydrometeorological conditions and watershed physiography as well as their interacting influences.

It is evident that there are three processes involved in a rainfall-runoff relationship:

(1) rainfall process

(2) transformation process

(3) runoff process

The quantification of this relationship depends on the two former processes, namely, rainfall process and transformation process. Their description is the cornerstone of classifying models proposed to predicate the rainfall-runoff relationships. For example, if the rainfall process is completely known apriori, as is often the case when dealing with an individual rainfall event, i.e., the rainfall process is deterministic then, with an assumed structure of the transformation process, the resulting model for the runoff process will be deterministic as well and will be able to describe the runoff hydrograph completely due to that known rainfall episode. However, if the rainfall process is described by a set of statistical parameters or its structure is assumed to be stochastic then the resulting model of the runoff process will also be stochastic. This latter approach is the one we are concerned with in the present investigation. Figure 1-1 shows evolution of three broad, general classes of models for the runoff process. This classification is merely illustrative and obviously not exhaustive.

1.2 OBJECTIVES

The present study develops explicit relationships between statistical parameters of rainfall and those of runoff, without making any assumptions regarding the probability distribution of rainfall. However, it is assumed that the rainfall process is random and the transformation process is characterized by (1) a linear function, and (2) a nonlinear function. Several studies (Matalas, 1963; Shen, 1965; Jeng and Yevjevich, 1966; Frind, 1969) have considered this problem in the past. The solutions developed here are different from those of Frind (1969); we point out the reasons for the difference between them.

By computer experimentation linear and nonlinear hypotheses regarding the transformation process are examined and compared. Analytical expressions for the distribution parameters of runoff process are derived in terms of those of rainfall process, utilizing linear hypotheses. However, for nonlinear hypotheses these expressions are not wieldly.

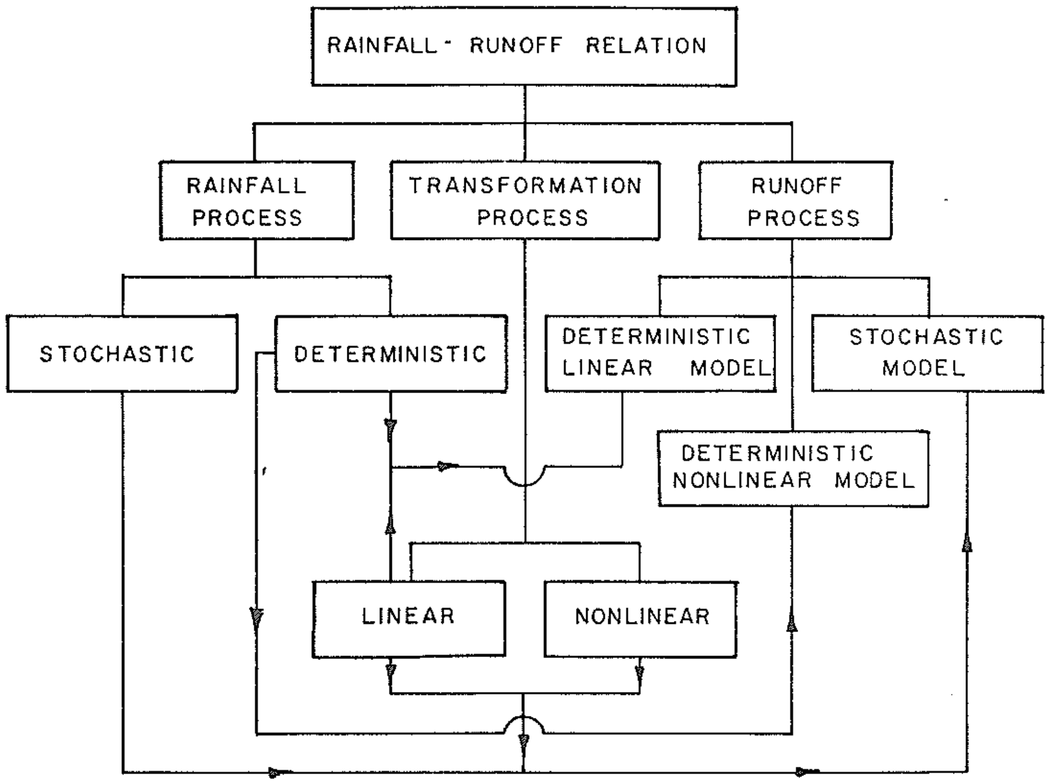

Fig. 1-1. Rainfall-runoff relation.

Chapter 2

LINEAR RAINFALL-RUNOFF RELATIONSHIP

2.1 RAINFALL PROCESS

Although rainfall is a continuous process (including periods of zero intensity), it is convenient, for computational purposes, to descretize it using the mean value of intensity over some time base. Commonly used time bases include the hour, the day, the month, or the year. The randomness of the rainfall process is influenced by the choice of this time base. For example, daily or monthly means are subject to cyclical influence of the seasons, resulting in some degree of interdependence among the successive terms in the series, but disagreement exists among researchers regarding it. A possible link between eleven-year sun spot cycle and rainfall has been suspected (Soucek, 1967); however, studies of Yevjevich (1966), and Rodriguez and Yevjevich (1967) found no evidence of such a link. In the present study we deal with annual rainfall and assume it to be pure-random. Part of this rainfall returns to the atmosphere by means of evaporation and evapotranspiration. Another part may recharge the aquifers extending beyond the boundaries of a given basin. The remainder of rainfall becomes eventually the basin runoff. That portion of rainfall which contributes entirely to runoff is usually referred to as effective rainfall. It is this rainfall that is input in the rainfall-runoff relationship. Henceforth, rainfall would imply effective rainfall.

2.2 TRANSFORMATION PROCESS

The transformation process encompasses virtually the entire domain of hydrologic cycle. Numerous processes, complex and interdependent, participate in this process. Despite its intractable complexity it is sometimes reasonable to approximate it by a linear reservoir which has been the basis of several investigations (Nash, 1957; Dooge, 1959; Chow, 1964) and more importantly the unit hydrograph theory (Sherman, 1932), a commonly accepted tool in applied hydrology. We first derive mathematical expressions for instantaneous runoff due to rainfall occurring uniformly throughout the year. It would be implicit in the derivation that rainfall would stop at the end of the year, and derived runoff would be entirely due to this rainfall and not due to any other rainfall that might occur in any other years. Second, mathematical expressions are derived for runoff volume during a given year due to rainfall in the same year and the preceding years.

Let j define the year for which we wish to obtain runoff due to rainfall that occurred in the (j - i)th year, where i defines the number of the years before the year j. Then we can write the continuity equation (see Fig. 2-1) as:

$$\frac{dS_{ji}}{dt} = 0 \qquad t \leq j - i \tag{2-1}$$

$$\frac{dS_{ji}}{dt} + q_{ji} = P_{j-i} \qquad j - i \leq t \leq j - i + 1 \tag{2-2}$$

$$\frac{dS_{ji}}{dt} + q_{ji} = 0 \qquad j - i + 1 \leq t \tag{2-3}$$

and the storage equation for a linear reservoir as:

$$q_{ji} = kS_{ji} \tag{2-4}$$

where

P_{j-i} = rainfall intensity for (j - i)th year,

q_{ji} = instantaneous discharge due to P_{j-i},

S_{ji} = basin storage due to P_{j-i},

k = storage coefficient, and

t = time.

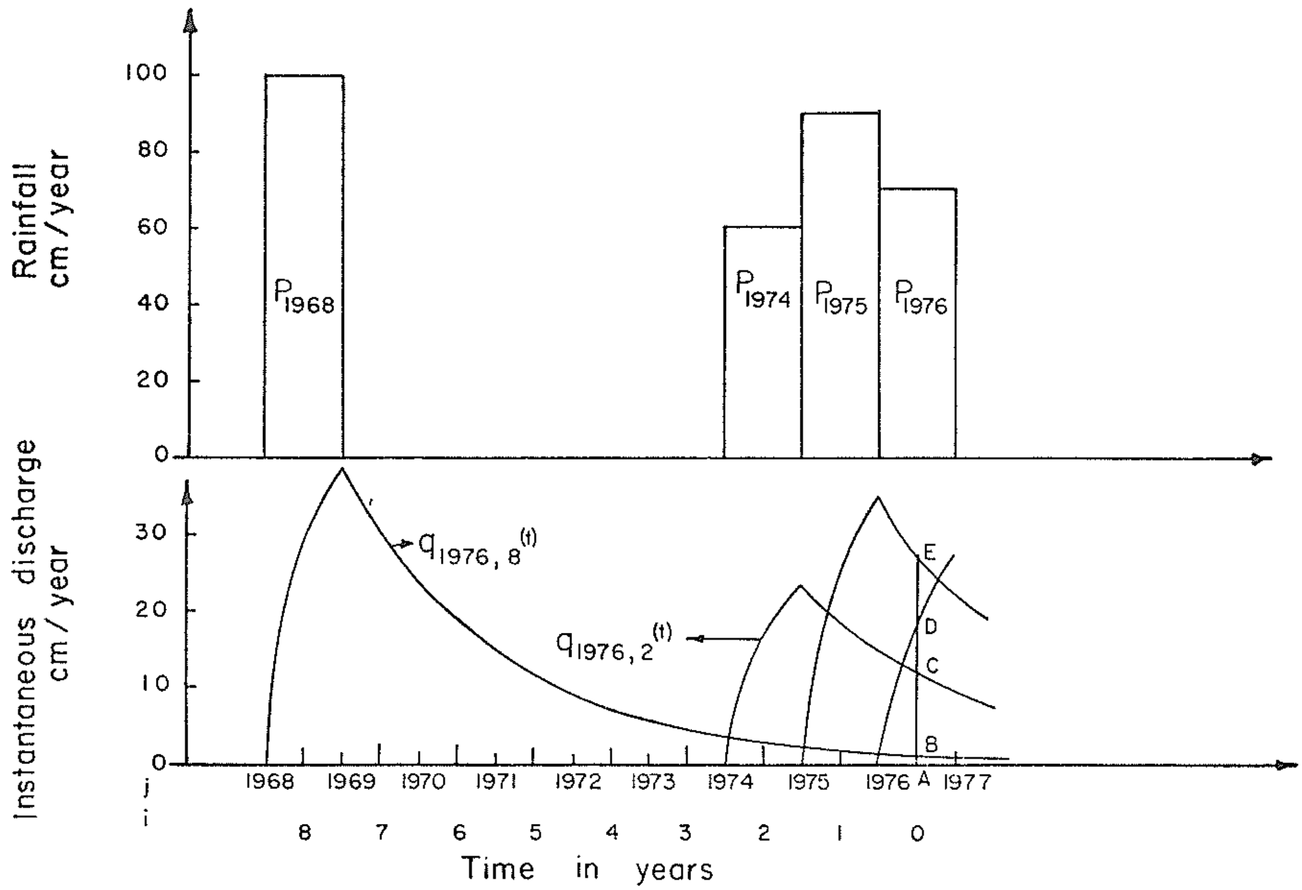

Fig. 2-1. Graphical representation of rainfall-runoff relation. AB, AC, AD and AE represent instantaneous discharges at time t (where 1976 < t < 1977) due to rainfalls in 1968, 1974, 1976 and 1975. Note that discharge due to rainfall in 1975 is greater than discharge due to rainfall in 1976.

The solution of Eqs. (2-2) and (2-4) with the initial condition $q_{ji} = 0$ when $t = j - i$ is:

$$q_{ji}(t) = P_{j-1}[1 - e^{-k(t+i-j)}] \qquad j - i \leq t \leq j - i + 1 \qquad (2\text{-}5)$$

The initial condition here does not necessarily imply a dry surface; it merely says that at the start of the year (j - i), $t = j - i$, q_{ji} due to P_{j-i} is 0. This condition is also implied by Eqs. (2-1) and (2-4). q_{ji}, as a function of time, in the subsequent years is given by the solution of Eqs. (2-3) and (2-4) as:

$$q_{ji} = P_{j-i}(1 - e^{-k})\, e^{-k(t-j+i-1)} \qquad j - i + 1 \leq t \qquad (2\text{-}6)$$

The initial condition here is $q_{ji} = P_{j-1}(1 - e^{-k})$ when $t = j - i + 1$. Thus we can summarize the solution of Eqs. (2-1) - (2-4) as:

$$q_{ji}(t) = \begin{cases} 0 & t \leq j - i \\ P_{j-i}[1 - e^{-k(t+i-j)}] & j - i \leq t \leq j - i + 1 \\ P_{j-i}(1 - e^{-k})\, e^{-k(t-j+i-1)} & j - i + 1 \leq t \end{cases} \qquad (2\text{-}7)$$

Our interest is, however, in the determination of q during a given year j due to all rainfall episodes that occurred during and prior to the year j. We call this quantity as q_j. Then we can write (see Fig. 2-1):

$$q_j(t) = \sum_{i=0}^{\infty} q_{ji}(t) \qquad \text{where } j \leq t \leq j + 1 \qquad (2\text{-}8)$$

The subscript i denotes a past i^{th} event that occurred before the year j. Thus Eq. (2-8) becomes:

$$q_j(t) = P_j(1 - e^{-k(t-j)}) + (1 - e^{-k}) \sum_{i=1}^{\infty} P_{j-i}\, e^{-k(t-j+i-1)} \qquad j \leq t \leq j + 1 \qquad (2\text{-}9)$$

Equation (2-9) gives instantaneous discharge during the course of the year j at $j \leq t \leq j + 1$. The total discharge Q at the end of the year j can be obtained by integrating Eq. (2-9) from $t = j$ to $t = j + 1$:

$$Q_j = P_j[1 + (e^{-k} - 1)/k] + \frac{(1 - e^{-k})^2}{k} \sum_{i=1}^{\infty} P_{j-i}\, e^{-k(i-1)} \tag{2-10}$$

For simplicity let us define:

$$\alpha_0 = 1 + (e^{-k} - 1)/k \tag{2-11}$$

$$\alpha_i = \frac{(1 - e^{-k})^2}{k}\, e^{-k(i-1)} \qquad i = 1, 2, \ldots \tag{2-12}$$

Equation (2-10) can then be written as:

$$Q_j = \sum_{i=0}^{\infty} \alpha_i\, P_{j-i} \tag{2-13}$$

The coefficients α_i are the weighting coefficients possessing the following properties:

$$\text{(a)} \quad \sum_{i=0}^{\infty} \alpha_i = 1 \tag{2-14a}$$

$$\text{(b)} \quad 0 < \alpha_i < 1 \qquad i = 0, 1, 2, \ldots \tag{2-14b}$$

$$\text{(c)} \quad \alpha_1 > \alpha_2 > \alpha_3 \ldots > \alpha_i > \alpha_{i+1} \ldots \tag{2-14c}$$

Equation (2-13) gives the runoff volume as a moving average of rainfall. Properties of α given by Eqs. (2-14a) - (2-14c) are identical to those given by Matalas (1963) without proof, with the exception that inequality (2-14c) also includes α_0 where $\alpha_0 > \alpha_1$. From Eqs. (2-11) and (2-12) it can be seen that this condition given by Matalas (1963) may not be true, and depends only on the value of k. For example, if $k = 0.9625$ then $\alpha_0 = 0.3579$ and $\alpha_1 = 0.3696$, for which $\alpha_0 < \alpha_1$. If k has a smaller value then α with a higher subscript may be larger than α_0. This result is more compatible with physical realism than the one given by Matalas (1963), since in some watersheds water may take an extended period of time to reach the outlet such that the main contribution to runoff may not be from the rainfall occurring at that time.

2.2.1 Length of Carryover

The first term of Eq. (2-10) represents the contribution of current rainfall to runoff and the second term the contribution of rainfall in previous years. That portion of runoff which is due to rainfall in the past years is usually termed as carryover. To determine the length of carryover Eq. (2-13) can be written as:

$$Q_j = \sum_{i=0}^{n} \alpha_i P_{j-1} + \sum_{i=n+1}^{\infty} \alpha_i P_{j-1} \qquad (2\text{-}15)$$

where n is the length of the carryover. The second term in Eq. (2-15) is tantamount to the actual error e_a incurred in the calculation of Q, since Eq. (2-13) must be terminated at some point. Thus, it is clear that the actual error is not only a function of basin characteristics but also a function of the complete previous rainfall history of the basin. That is,

$$e_a = e(k, P_{n+1}, P_{n+2}, \ldots)$$

$$= \sum_{i=n+1}^{\infty} \alpha_i P_{j-i} \qquad (2\text{-}16)$$

Obviously, determination of e_a requires the knowledge of complete previous rainfall history of the basin which is never known. To overcome this difficulty, let us define an error e_p associated with the maximum observed rainfall intensity, P_{max}, such that:

$$e_p = e(k, P_{max})$$

$$= P_{max} \sum_{i=n+1}^{\infty} \alpha_i \qquad (2\text{-}17)$$

It is then understood that:

$$e_a \leq e_p = \frac{P_{max} (1 - e^{-k}) e^{-kn}}{k} \tag{2-18}$$

Then we can obtain:

$$n = \frac{1}{k} \ln \frac{P_{max}(1 - e^{-k})}{k e_p} \tag{2-19}$$

which is the length of the carryover. If it is assumed that k is known without any error then the only source of error (from Eq. (2-13)) is rainfall. If ΔP is defined as the absolute error in the estimation of mean areal rainfall then the resulting error in discharge ΔQ can be written as:

$$\Delta Q_j = \sum_{i=0}^{\infty} \alpha_i \Delta P_{j-1} \tag{2-20}$$

Since the distribution of ΔP_{j-1} may not be known, it is more realistic to write:

$$\Delta Q_j = \sum_{i=0}^{\infty} \alpha_i \Delta P \tag{2-21}$$

where ΔP is absolute value of the maximum possible error in calculating average areal rainfall. Then we can simply write:

$$\Delta Q_j = \Delta P \sum_{i=0}^{\infty} \alpha_i \tag{2-22}$$

Using Eq. (2-14a),

$$\Delta Q_j = \Delta P \tag{2-23}$$

We can now simply write:

$$e_p = \Delta P \tag{2-24}$$

The value of e_p estimated from Eq. (2-24) should be used in Eq. (2-19) to evaluate n.

2.2.2 Comparison with Frind's Solution

It is now appropriate to compare our solution (Eq. (2-10)) with that of Frind (1969; Eq. (12)). If we take t = j + 1, which represents the end of the j^{th} year, Eq. (2-9) yields:

$$q_j = (1 - \bar{e}^{k}) \sum_{j=0}^{\infty} P_{j-1} \bar{e}^{ki} \qquad (2\text{-}23)$$

which is identical to the equation given by Frind (1969). This shows that his equation gives instantaneous discharge at the end of the current year rather than the total discharge. This leads us to believe that an implicit assumption in the derivation by Frind (1969) is that the total discharge in a given year is equal to the instantaneous discharge at the end of that year. Some obvious observations follow from his derivation:

(1) Overprediction of q during the year of rainfall.

(2) Underprediction of q during years subsequent to the year of rainfall.

(3) As the new terms added to Eq. (2-25) (or Eq. 12 of Frind) underprediction tends to cancel overprediction (see Fig. 2-2).

(4) The behavior of Eq. (2-25) increases the length of carryover unnecessairly; Eq. (2-10) converges much faster. That is probably why Frind took carryover four times the computed carryover.

For computational purposes e_p was fixed at 0.001 cm/year and P at 100 cm/year. It is clear from Eq. (2-19) that n depends on k. Henceforth we will assume that there is a unique correspondence between n and k; that is, as k increases, n decreases accordingly by virtue of Eq. (2-19) and viceversa.

2.3 RUNOFF PROCESS

The mean annual runoff series, obtained from the basin, by transformation of the rainfall process will possess some statistical distribution

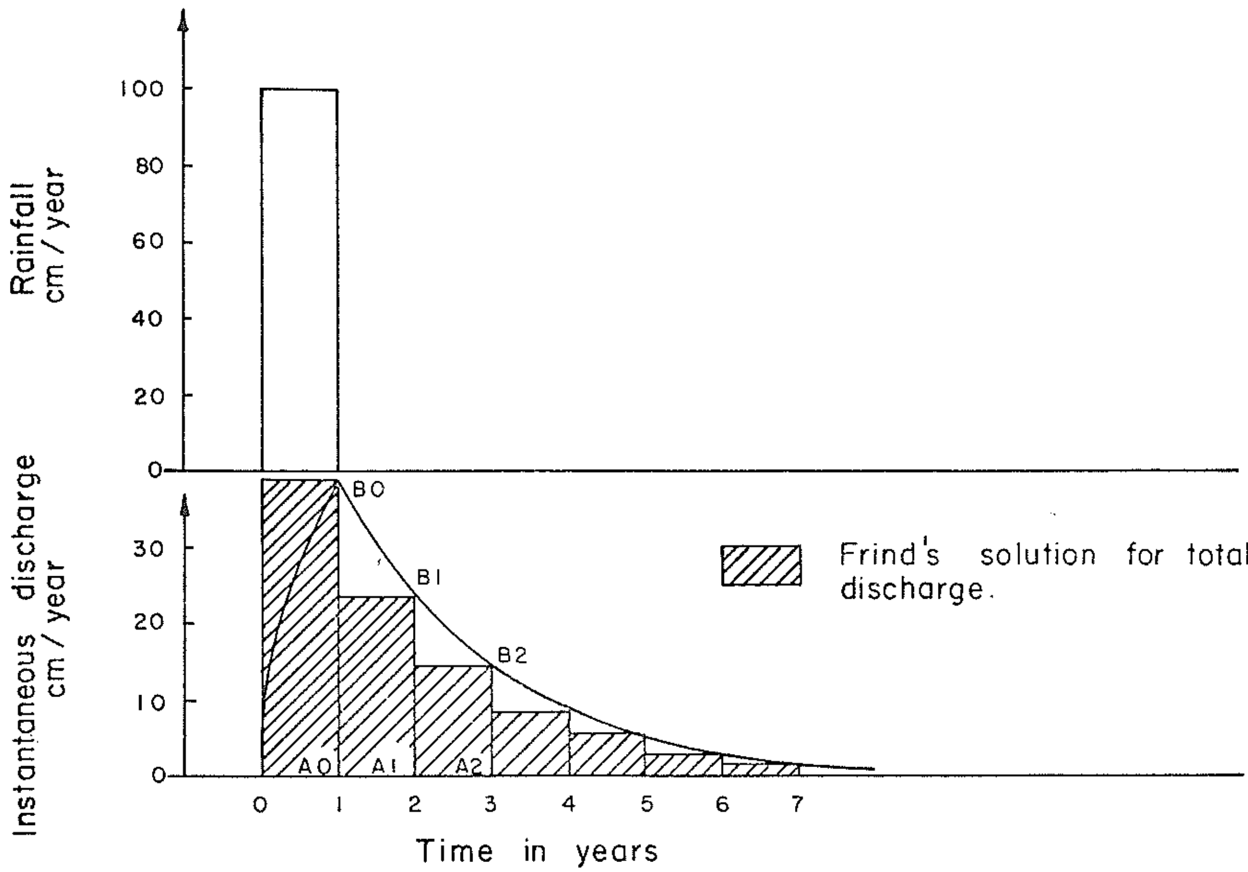

Fig. 2-2. Graphical representation of Frind's solution.

that we want to investigate. As yet we do not know how its distribution properties are in relation to those of the rainfall process.

2.3.1 Mean, μ

Using Eq. (2-13) the expected value of Q can be written as:

$$E[Q_j] = E\left[\sum_{i=0}^{\infty} \alpha_i P_{j-i}\right]$$

$$= \sum_{i=0}^{\infty} \alpha_i E[P_{j-i}] \qquad (2\text{-}26)$$

where E is the expectation operator. Assuming $[P_{j-i}]$ to be independent, identically distributed, we can write:

$$E[Q_j] = E[P_j] \sum_{i=0}^{\infty} \alpha_i \qquad (2\text{-}27)$$

Since $E[Q_j]$ and $E[P_j]$ are not dependent on j,

$$\bar{Q} = \bar{P} \sum_{i=0}^{\infty} \alpha_i \qquad (2\text{-}28)$$

where $\bar{Q}$ is the mean of Q and $\bar{P}$ the mean of P. Using Eq. (2-14a) we obtain:

$$\bar{Q} = \bar{P} \qquad (2\text{-}29)$$

or we can write:

$$\mu_Q = \mu_P$$

Thus in the linear transformation mean of the runoff process is the same as the mean of the rainfall process irrespective of its probability distribution.

This same conclusion can also be deduced directly from Eq. (2-2). Writing the continuity equation without subscripts,

$$\frac{dS}{dt} + q = P \qquad (2\text{-}30)$$

Taking the expectation of Eq. (2-30),

$$E\left[\frac{dS}{dt}\right] + E[q] = E[P] \qquad (2\text{-}31)$$

We can then write:

$$\frac{d}{dt}\, E[S] + \bar{q} = \bar{P} \qquad (2\text{-}32)$$

which simply yields:

$$\bar{q} = \bar{P} \qquad (2\text{-}33)$$

In Eq. (2-30) q is the total outflow and P the total inflow. Thus q is identical to Q, and Eq. (2-33) is the same as Eq. (2-29). The result of Eq. (2-29) is consistent with physical intuition because in the long run all rainfall must become runoff.

2.3.2 Variance, σ^2

Variance of Q can be written as:

$$\sigma_Q{}^2 = E[(Q - \bar{Q})^2] \qquad (2\text{-}34)$$

Using Eq. (2-13) and (2-26) we obtain:

$$\sigma_Q{}^2 = E\left[\left\{\sum_{i=0}^{\infty} \alpha_i\, P_{j-i} - \bar{P} \sum_{i=0}^{\infty} \alpha_i\right\}^2\right]$$

$$= E\left[\left\{\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})\, \alpha_i\right\}^2\right]$$

Expanding the quadratic term within the braces,

$$\sigma_Q{}^2 = E\left[\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})^2\, \alpha_i{}^2 + 2 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} (P_{j-i} - \bar{P})(P_{j-r} - \bar{P})\, \alpha_i\, \alpha_r\right]$$

Taking the expectation operator inside the summation,

$$\sigma_Q{}^2 = \sum_{i=0}^{\infty} \alpha_i{}^2\, E(P_{j-i} - \bar{P})^2 + 2 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i\, \alpha_r\, E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})] \qquad (2\text{-}35)$$

If we assume that $[P_{j-i}]$ and $[P_{j-r}]$ are independent, identically distributed then Eq. (2-35) can be written as:

$$\sigma_Q^2 = E(P - \bar{P})^2 \sum_{j=0}^{\infty} \alpha_j^2 + 2E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})] \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i \alpha_r \qquad (2\text{-}36)$$

We can define covariance and variance of P_{j-i} as:

$$\mathrm{Cov}\,(P_{j-i}, P_{j-r}) = E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})] = 0 \qquad \text{if} \qquad i \neq r$$

and

$$\sigma_P^2 = E(P_{j-i} - \bar{P})^2 \qquad \text{if} \qquad i = r$$

Then Eq. (2-36) becomes:

$$\sigma_Q^2 = \sigma_P^2 \sum_{i=0}^{\infty} \alpha_i^2$$

Thus we can write:

$$\frac{\sigma_Q^2}{\sigma_P^2} = \sum_{i=0}^{\infty} \alpha_i^2 \qquad (2\text{-}37)$$

Equation (2-37) implies that the ratio of the variances of runoff and rainfall is constant. It also implies that the variance of runoff is less than the variance of rainfall and thus supports the finding of Shen (1965) for a particular case of lognormal probability distribution. To prove it we need to show that:

$$\sum_{i=0}^{\infty} \alpha_i^2 < 1 \qquad (2\text{-}38)$$

We know from Eqs. (2-14a) and (2-14b) that

$$\sum_{i=0}^{\infty} \alpha_i = 1 \qquad \text{and} \qquad 0 < \alpha_i < 1 \qquad (2\text{-}39)$$

Then it is obvious that

$$\alpha_i^2 < \alpha_i$$

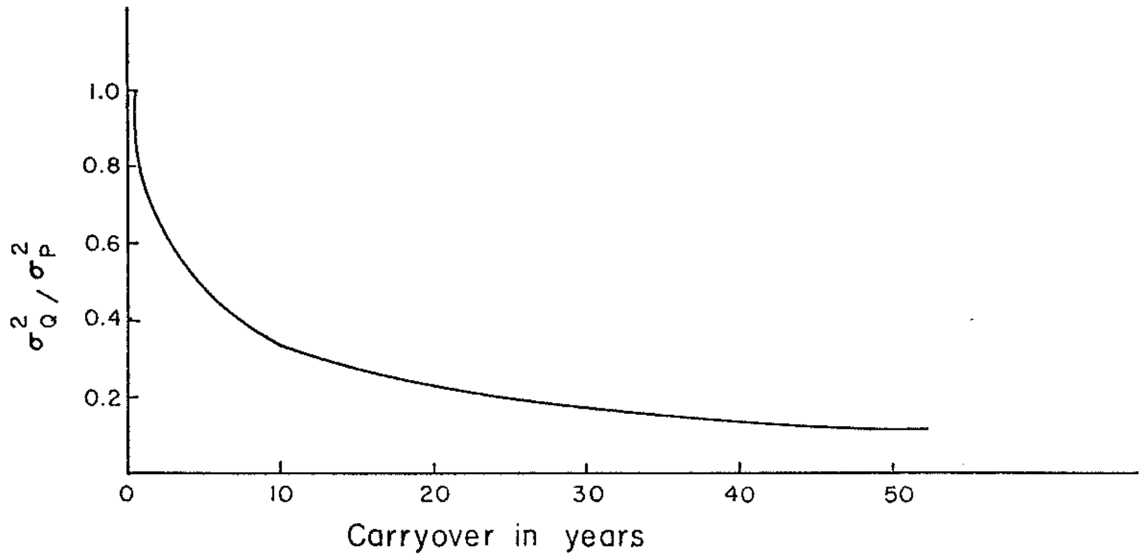

Fig. 2-3. Ratio of variances as a function of carryover.

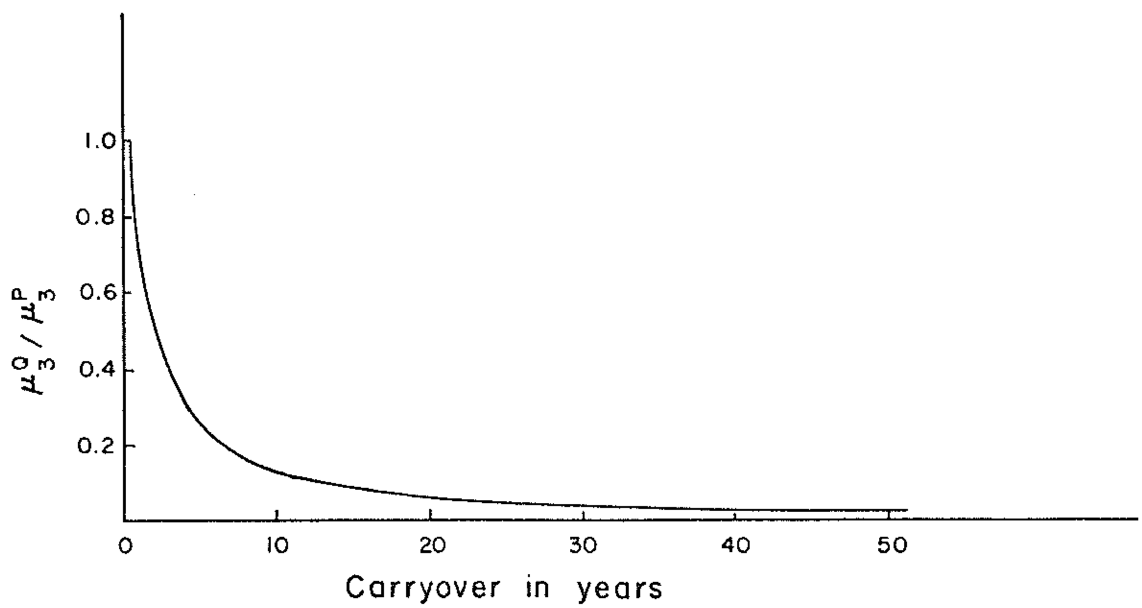

Fig. 2-4. Ratio of third moments as a function of carryover.

and hence

$$\sum_{i=0}^{\infty} \alpha_i^{\,2} < \sum_{i=0}^{\infty} \alpha_i = 1$$

The value of $\alpha_i^{\,2}$ will depend on k and n and so will the value of $\sum\alpha_i^{\,2}$. Let us now write the expressions for standard deviation σ and coefficient of variation Cv:

$$\sigma_Q = \sigma_P \left[\sum_{i=0}^{\infty} \alpha_i^{\,2}\right]^{0.5} \tag{2-40}$$

$$Cv_Q = \frac{\sigma_P}{\bar{P}} \left[\sum_{i=0}^{\infty} \alpha_i^{\,2}\right]^{0.5} \tag{2-41}$$

The ratio of variances given by Eq. (2-37) was plotted for different values of the length of carryover n as shown in Fig. 2-3. It is clear that this ratio decreases with increasing n tending asymptotically to zero; it should be clear from Eq. (2-14) too. Furthermore, this ratio is independent of the type of rainfall distribution.

2.3.3 Third Moment about Mean, μ_3

The third moment of runoff μ_3 about the mean can be written as:

$$\mu_3^{\,Q} = E[Q - \bar{Q}] = E\left[\left\{\sum_{i=0}^{\infty} \alpha_i P_{j-1} - \bar{P} \sum_{i=0}^{\infty} \alpha_i\right\}^3\right] \tag{2-42}$$

$$= E\left[\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})\,\alpha_i\right]^3$$

We can expand and write it as:

$$\mu_3^{\,Q} = E\left[\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})^3 \alpha_i^{\,3} + 3 \sum_{\substack{i=0 \\ i\neq r}}^{\infty} \sum_{r=0}^{\infty} (P_{j-i} - \bar{P})(P_{j-r} - \bar{P})\, \alpha_i^{\,2}\, \alpha_r \right.$$

$$\left. + 6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty} (P_{j-i} - \bar{P})(P_{j-r} - \bar{P})(P_{j-\ell} - \bar{P})\, \alpha_i\, \alpha_r\, \alpha_\ell \right] \tag{2-43}$$

Taking the expectation operator inside the summation,

$$\mu_3^Q = \sum_{i=0}^{\infty} \alpha_i^3 \, E(P_{j-i} - \bar{P})^3 + 3 \sum_{\substack{i=0 \\ i \neq r}}^{\infty} \sum_{r=0}^{\infty} \alpha_i^2 \, \alpha_r \, E[(P_{j-i} - \bar{P})^2 (P_{j-r} - \bar{P})]$$

$$+ 6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty} \alpha_i \, \alpha_r \, \alpha_\ell \, E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})(P_{j-\ell} - \bar{P})] \qquad (2\text{-}49)$$

If $[P_{j-i}]$, $[P_{j-r}]$ and $[P_{j-\ell}]$ are independent, identically distributed then the above equation can be written as:

$$\mu_3^Q = E[P - \bar{P}]^3 \sum_{i=0}^{\infty} \alpha_i^3 + 3 \, E[(P - \bar{P})^2 (P - \bar{P})] \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \alpha_i^2 \, \alpha_r$$

$$+ 6 \, E[(P - \bar{P})(P - \bar{P})(P - \bar{P})] \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty} \alpha_i \, \alpha_r \, \alpha_\ell \qquad (2\text{-}45)$$

Since $E[P - \bar{P}] = E[P] - \bar{P} = 0$, we obtain:

$$\mu_3^Q = E[P - \bar{P}]^3 \sum_{i=0}^{\infty} \alpha_i^3 = \mu_3^P \sum_{i=0}^{\infty} \alpha_i^3 \qquad (2\text{-}46)$$

Thus we have:

$$\frac{\mu_3^Q}{\mu_3^P} = \sum_{i=0}^{\infty} \alpha_i^3 \qquad (2\text{-}47)$$

where $\sum_{i=0}^{\infty} \alpha_i^3 < 1$ for the same reasons as for $\sum_{i=0}^{\infty} \alpha_i^2$.

The ratio of the third moments was plotted for different values of n as shown in Fig. 2-4. It is clear that this ratio decreases with increasing n, tending asymptotically to zero and is independent of the type of rainfall distribution.

2.3.4 Skewness

The coefficient of skewness is defined as:

$$\text{Shew} = \frac{\mu_3}{(\mu_2)^{0.5}} \tag{2-48}$$

where $\mu_2 = \sigma^2$

Then the coefficient of skewness of runoff is:

$$\text{Skew}^Q = \frac{\sum_{i=0}^{\infty} \alpha_i^{\,3}}{\left(\sum_{i=0}^{\infty} \alpha_i^{\,2}\right)^{1.5}} \; \frac{\mu_3^{\,P}}{(\mu_2)^{1.5}} = \beta \; \text{Skew}^P \tag{2-49}$$

where

$$\beta = \frac{\sum_{i=0}^{\infty} \alpha_i^{\,3}}{\left(\sum_{i=0}^{\infty} \alpha_i^{\,2}\right)^{1.5}}$$

Thus we get:

$$\frac{\text{Skew}^Q}{\text{Skew}^P} = \beta \tag{2-50}$$

under the prescribed constraints on α_i, $\beta < 1$; we, however, need to show that:

$$\sum_{i=0}^{\infty} \alpha_i^{\,3} < \left(\sum_{i=0}^{\infty} \alpha_i\right)^{1.5}$$

we can write as:

$$\left(\sum_{i=0}^{\infty} \alpha_i^{\,2}\right)^{1.5} = \sum_{i=0}^{\infty} \alpha_i^{\,3} + \text{PROD} \tag{2-51}$$

where PROD > 0 and will consist of product terms. Thus it is obvious that $\beta < 1$.

From Eq. (2-50) it is clear that the skewness of runoff will be smaller than the skewness of rainfall regardless of its distribution. This is a mathematical confirmation of the findings of Shen (1965). The quantity β was plotted against the length of carryover as shown in Fig. 2-5. It is clear that β decreases as n increases, tending asymptotically to zero.

2.3.5 Fourth Moment about the Mean, μ_4

The fourth moment of runoff about its mean can be expressed as:

$$\mu_4^Q = E[Q - \bar{Q}]^4 \tag{2-52}$$

Substituting for Q and $\bar{Q}$,

$$\mu_4^Q = E\left(\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})\ \alpha_i\right)^4 \tag{2-53}$$

Expanding the right-hand side,

$$\mu_4^Q = E\left(\sum_{i=0}^{\infty} (P_{j-i} - \bar{P})^4\ \alpha_i^4 + 4 \sum_{\substack{i=0 \\ i \neq r}}^{\infty} \sum_{r=0}^{\infty} (P_{j-i} - \bar{P})^3 (P_{j-r} - \bar{P})\ \alpha_j^3\ \alpha_r \right.$$

$$+ 6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} (P_{j-i} - \bar{P})^2 (P_{j-r} - \bar{P})^2\ \alpha_i^2\ \alpha_r^2 + 24 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty} \sum_{s=\ell+1}^{\infty}$$

$$\left. (P_{j-i} - \bar{P})(P_{j-r} - \bar{P})(P_{j-\ell} - \bar{P})(P_{j-s} - \bar{P})\ \alpha_i\ \alpha_r\ \alpha_\ell\ \alpha_s \right) \tag{2-54}$$

Taking the expectation operator inside the summation,

$$\mu_4^Q = \sum_{i=0}^{\infty} \alpha_i^4\ E[P_{j-i} - \bar{P}]^4 + 4 \sum_{\substack{i=0 \\ i \neq r}}^{\infty} \sum_{r=0}^{\infty} \alpha_i^3\ \alpha_r\ E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})]$$

$$+ 6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i^2\ \alpha_r^2\ E[(P_{j-i} - \bar{P})^2 (P_{j-r} - \bar{P})^2] + 24 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty}$$

$$\sum_{s=\ell+1}^{\infty} \alpha_i\ \alpha_r\ \alpha_\ell\ \alpha_s\ E[(P_{j-i} - \bar{P})(P_{j-r} - \bar{P})(P_{j-\ell} - \bar{P})(P_{j-s} - \bar{P})] \tag{2-55}$$

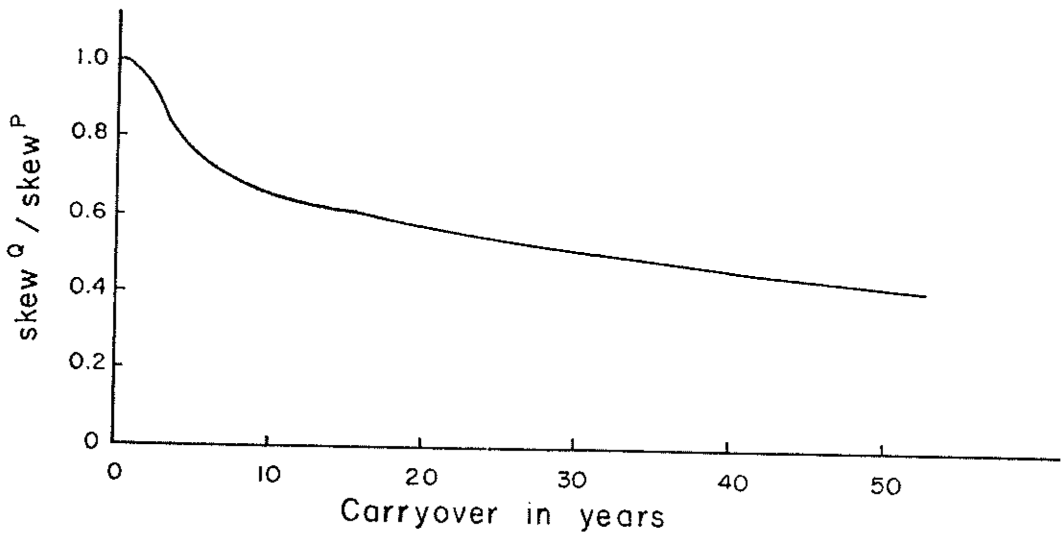

Fig. 2-5. Ratio of skewnesses as a function of carryover.

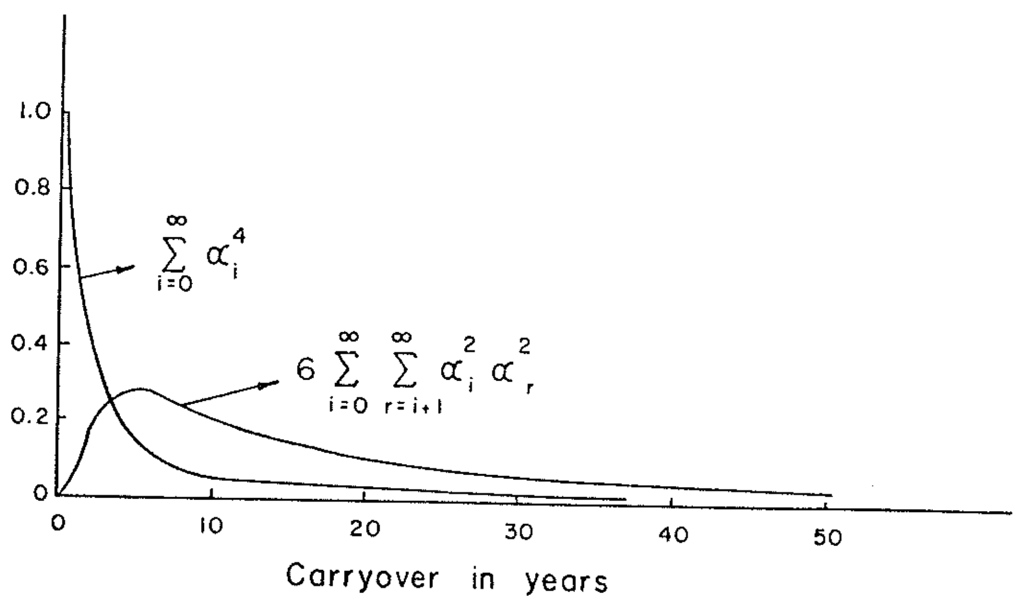

Fig. 2-6. Terms of fourth moment as a function of carryover.

Invoking the assumption of independence and identicality of sequences of P,

$$\mu_4{}^Q = \sum_{i=0}^{\infty} \alpha_i{}^4 \, E[P_{j-i} - \bar{P}]^4 + 4 \sum_{\substack{i=0 \\ i \neq r}}^{\infty} \sum_{r=0}^{\infty} \alpha_i{}^3 \, \alpha_r \, E[P_{j-i} - \bar{P}] \, E[P_{j-r} - \bar{P}]$$

$$+ 6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i{}^2 \, \alpha_r{}^2 \, E[P_{j-i} - \bar{P}]^2 \, E[P_{j-r} - \bar{P}]^2 + 24 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \sum_{\ell=r+1}^{\infty}$$

$$\sum_{s=\ell+1}^{\infty} \alpha_i \, \alpha_r \, \alpha_\ell \, \alpha_s \, E[P_{j-i} - \bar{P}] \, E[P_{j-r} - \bar{P}] \, E[P_{j-\ell} - \bar{P}] \, E[P_{j-s} - \bar{P}] \quad (2\text{-}56)$$

It was shown previously that $E[P - \bar{P}] = 0$. The second and third terms will, therefore, become zero. We can then write:

$$\mu_4{}^Q = E[P - \bar{P}]^4 \sum_{i=0}^{\infty} \alpha_i{}^4 + 6\{E[P - \bar{P}]^2\}^2 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i{}^2 \, \alpha_r{}^2 \quad (2\text{-}57)$$

By definition,

$$\mu_4{}^P = E[P - \bar{P}]^4$$

$$\sigma_P{}^2 = E[P - \bar{P}]^2$$

Then we obtain:

$$\mu_4{}^Q = \mu_4{}^P \sum_{i=0}^{\infty} \alpha_i{}^4 + 6 \, \sigma_P{}^2 \, \sigma_P{}^2 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i{}^2 \, \alpha_r{}^2 \quad (2\text{-}58)$$

The terms $\sum_{i=0}^{\infty} \alpha_i{}^4$ and $\sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i{}^2 \, \alpha_r{}^2$ we plotted against the length of carryover as shown in Fig. 2-6. It is clear that these quantities approach zero for large n.

2.3.6 Kurtosis, Kur

The kurtosis is defined as:

$$\text{Kur} = \frac{\mu_4}{(\mu_2)^2} \quad (2\text{-}59)$$

Then the kurtosis of runoff can be expressed as:

$$\frac{\mu_4^Q}{(\mu_2^Q)^2} = \frac{\mu_4^P \sum_{i=0}^{\infty} \alpha_i^4 + 6\,(\mu_2^P)^2 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i^2\,\alpha_r^2}{(\mu_2^P)^2 \left(\sum_{i=0}^{\infty} \alpha_i^2\right)^2}$$

$$= \frac{\mu_4^P}{(\mu_2^P)^2} \frac{\sum_{i=0}^{\infty} \alpha_i^4}{\left(\sum_{i=0}^{\infty} \alpha_i^2\right)^2} + 6\, \frac{\sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i^2\,\alpha_r^2}{\left(\sum_{i=0}^{\infty} \alpha_i\right)^2} \qquad (2\text{-}60)$$

We can simply write:

$$\text{Kur}^Q = a\ \text{Kur}^P + b \qquad (2\text{-}61)$$

where

$$a = \frac{\sum_{i=0}^{\infty} \alpha_i^4}{\left(\sum_{i=0}^{\infty} \alpha_i^2\right)^2}$$

$$b = \frac{6 \sum_{i=0}^{\infty} \sum_{r=i+1}^{\infty} \alpha_i^2\,\alpha_r^2}{\left(\sum_{i=0}^{\infty} \alpha_i^2\right)^2}$$

The kurtosis of Q was plotted against the length of carryover for three different types of rainfall distributions including uniform, triangular and trapezoidal. The plot is shown in Fig. 2-7. It is evident that as n increases, Kur^Q approaches a value of 3.

2.3.7 Distribution Characteristics

From the previous discussion it is evident that:

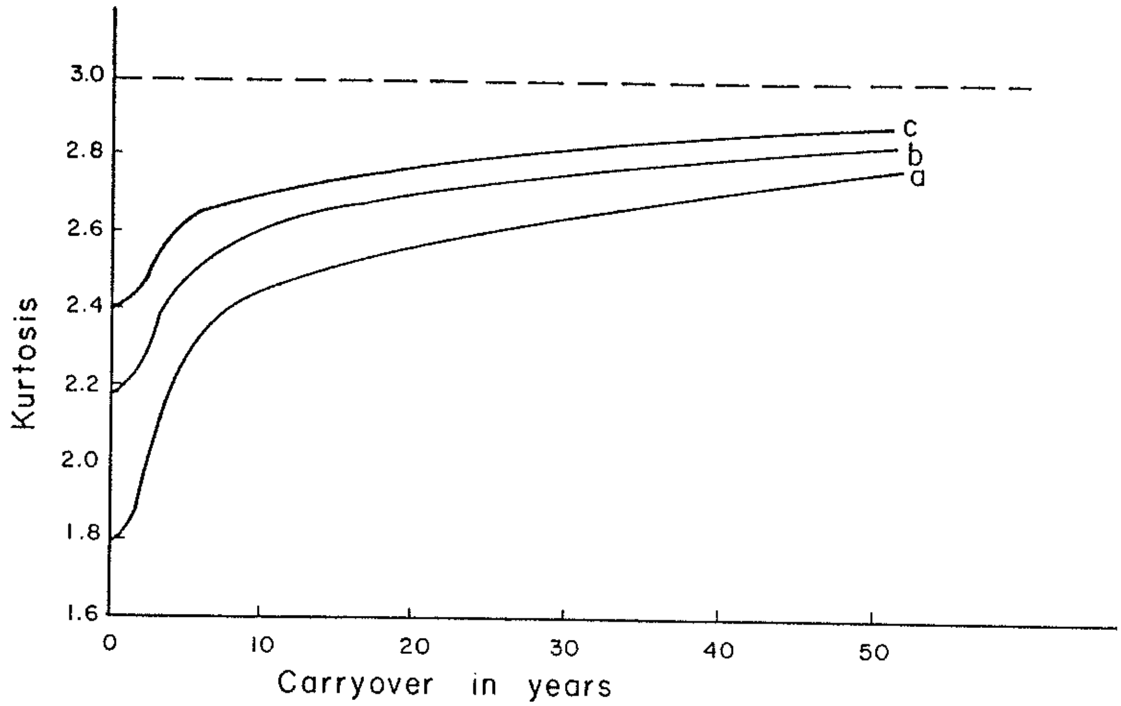

Fig. 2-7. Kurtosis as a function of carryover; rainfall has (a) uniform distribution, (b) trapezoidal distribution, (c) triangular distribution.

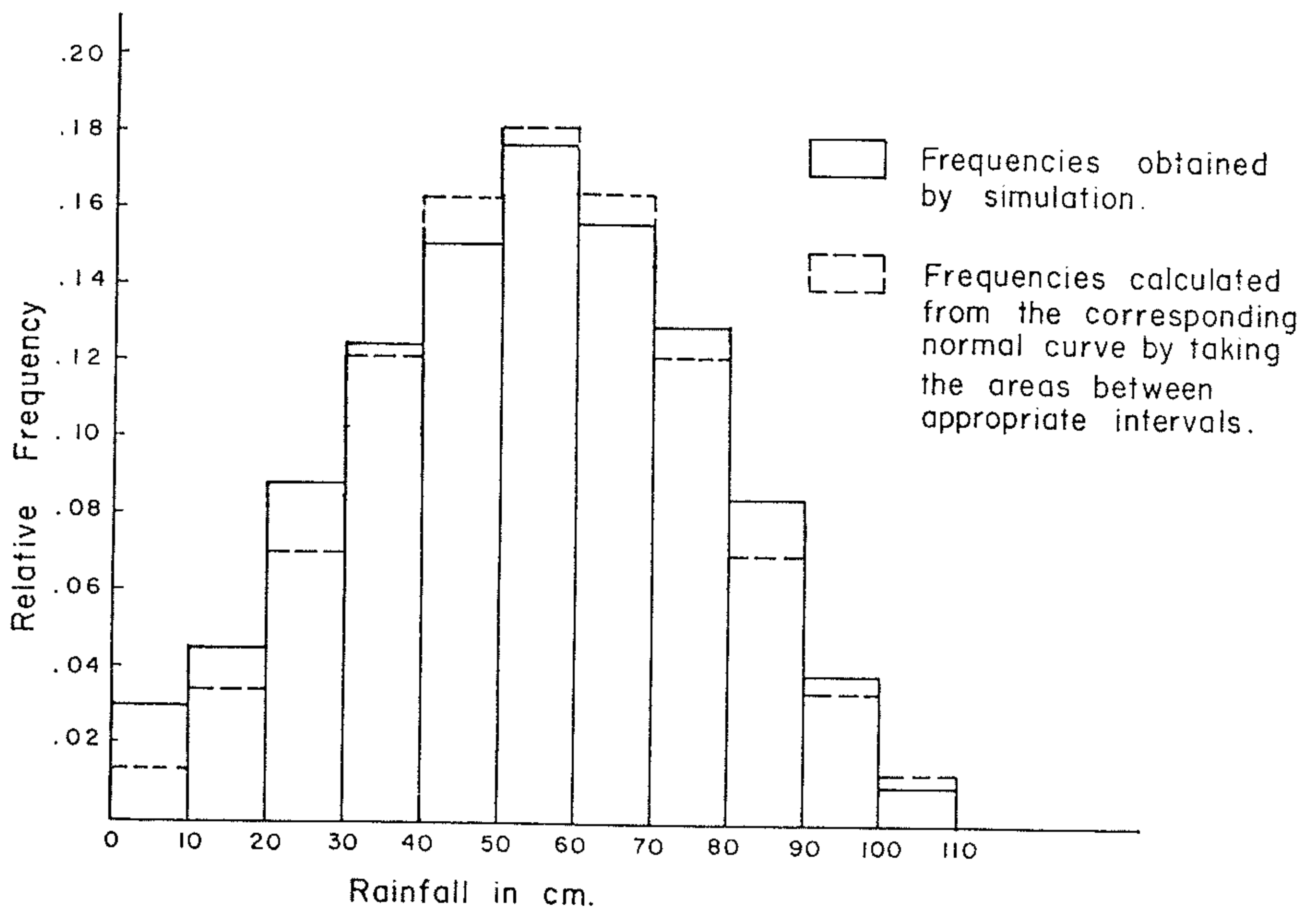

Fig. 2-8. Simulation of runoff distribution.

1. $\lim_{n\to\infty} \mu_i^Q = 0 \qquad i = 2, 3, 4, \ldots$

2. $\lim_{n\to\infty} \beta = 0$

3. $\lim_{n\to\infty} a = 0$

4. $\lim_{n\to\infty} b = 3$

Thus if $n\to\infty$, the statistical distribution of Q will have the following characteristics:

a. The second, third and higher moments around the mean go to zero. The higher moments will go to zero faster than the lower moments.

b. Skewness approaches zero.

c. Kurtosis approaches three.

These characteristics suggest that the distribution of Q approaches a normal distribution as n approaches infinity regardless of the distribution of P.

2.3.8 Simulation of Runoff

The hypotheses of the foregoing sections were tested by simulation of runoff distribution based on Eq. (2-13) in accordance with an assumed distribution of rainfall. Maximum observed rainfall intensity P_{max}, error e_p and storage coefficient k were chosen to be 100 cm/year, 0.001 cm/year and 2 $year^{-1}$ respectively. To calculate the length of carryover Eq. (2-19) was modified to be:

$$n = \frac{1}{k} \ln \frac{P_{max}(1 - e^{-k})}{k\, e_p} + 3 \qquad (2\text{-}62)$$

to account for (a) inadmissibility of zero subscript in the computer, (b) roundoff errors and (c) safety. n was then obtained as 8 years. Consequently, only the first 8 terms of Eq. (2-13) were evaluated to calculate Q.

Rainfall P was assumed to have a uniform distribution:

$$f(P) = \begin{cases} \frac{1}{110} & 0 \leq P \leq 110 \\ 0 & \text{otherwise} \end{cases} \tag{2-63}$$

Eight rainfall intensities were randomly selected from the above distribution for the first 8 terms of Eq. (2-13) in order to generate one Q value. This same procedure, the selection of 8 random numbers from the same uniform distribution followed by their substitution into Eq. (2-13) to calculate one Q value, was repeated 1000 times, thus 1000 different Q values were generated. The number of Q values falling between each 10 cm interval, from 0 to 110 cm, was counted and divided by 1000 (the total number of Q's). Thus relative frequency distribution of Q was obtained. Figure 2-8 shows the relative frequency distribution of runoff in solid black lines.

To compare the above relative frequency distribution with a normal distribution Eqs. (2-29) and (2-37) were utilized to calculate the population mean and population variance of runoff respectively. Population mean and population variance of rainfall are 55 cm and 1008.3 cm^2 (from Eq. (2-63)) respectively.

$\sum_{i=0}^{\infty} \alpha_i^2$ is given as:

$$\sum_{i=0}^{\infty} \alpha_i^2 = \left(1 + \frac{e^{-k} - 1}{k}\right)^2 + \frac{(1 - e^{-k})^4}{k^2(1 - e^{-2k})}$$

For k = 2 the above equation yields:

$$\sum_{i=0}^{\infty} \alpha_i^2 = 0.4646$$

Substitution of the above values into Eqs. (2-29) and (2-37) yields:

$\mu = 55$ cm

$\sigma^2 = 468.47$ cm^2

where μ and σ^2 are population mean and variance of runoff respectively. The normal density function of Q, g(Q), can be defined as:

$$g(Q) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{Q-\mu}{\sigma}\right)^2\right\} \tag{2-64}$$

The areas under the above normal curve between each 10 cm interval, from 0 to 110 cm, were calculated. These calculated areas are shown in Fig. 2-8 with histograms drawn in dotted lines. It is evident that both histograms are in close agreement. Thus the probability distribution of runoff can be approximated with Eq. (2-64).

Figure 2-7 shows that if rainfall P were to have a traingular or trapezoidal distribution, then the resulting distribution of runoff will approach a normal distribution even faster. To generalize this convergence to a normal curve let us define

$w = |Kur^P - 3|$

Consider two input distribution $f_1(P)$ and $f_2(P)$ having w_1 and w_2 respectively. It can be shown that if $w_1 < w_2$, the distribution of Q obtained from $f_1(P)$ will approach a normal curve faster than the distribution of Q obtained from $f_2(P)$.

Chapter 3

NONLINEAR RAINFALL-RUNOFF RELATIONSHIP

In the previous chapter the transformation process was assumed to be linear. In this chapter we investigate the rainfall-runoff relationship, utilizing a nonlinear transormation process. The assumptions regarding the pure randomness of the rainfall process will be retained. The notations of the previous chapter will be used, although their meaning and dimensionality might change due to nonlinearity.

3.1 TRANSFORMATION PROCESS

The continuity equation is given by

$$\frac{dS}{dt} + q = P_i \qquad i = j\text{-}m, j \quad ; \quad j \geqq i \tag{3-1}$$

and the storage equation for a nonlinear reservoir is

$$q = KS^n \qquad n > 1 \tag{3-2}$$

where

S = instantaneous storage duirng the year i,

q = instantaneous discharge during the year i,

P_i = rainfall intensity in the year i,

n = a dimensionless parameter, an index of nonlinearity,

K = storage coefficient,

m = length of carryover, and

j = the year in which runoff is desired.

Combining Eqs. (3-1) and (3-2) we obtain:

$$\frac{d}{dt}(KS^n) + q = P_i \tag{3-3}$$

Equation (3-3) is the basic differential equation for a single nonlinear storage element. Explicit analytical solutions of Eq. (3-3) are tractable for integral values of n; however, numerical solutions would be the only resort for non-integer values of n. Without loss of generality the rainfall-runoff problem can be studied for integer values of n. In this study n is taken as 2 and based on this some general conclusions are reached. For n = 2 the solution of Eq. (3-3) is

$$S_{i+1} = \sqrt{\frac{P_i}{K}}\ \frac{1-C_i \exp(-2\sqrt{KP_i})}{1+C_i \exp(-2\sqrt{KP_i})} \qquad i = j-m,\ j \tag{3-4}$$

where C_i is integration constant during the year i and is given by

$$C_i = \frac{\sqrt{P_i} - \sqrt{K}\, S_i}{\sqrt{P_i} + \sqrt{K}\, S_i} \qquad i = j-m,\ j \tag{3-5}$$

and S_i and S_{i+1} are the instantaneous storages at the beginning and the end of the year i. We note that Eqs. (3-4) and (3-5) are recurrence relationships. That is, if S_{j-m} is known, S_{j-m+1} can be calculated; similarly if S_{j-m+1} is known, S_{j-m+2} can be calculated and so on and so forth. Once S_{j+1} is obtained, the total discharge Q_j in the jth year is given by

$$Q_j = P_j + S_j - S_{j+1} \tag{3-6}$$

In this study S_{j-m} is taken to be zero and m is then defined as:

$$S_u(P_{max}) - S_{j+1}(P_{max}) \leqq \varepsilon \tag{3-7}$$

where $S_u(P_{max})$ is steadystate solution of Eq. (3-3) for P_i as constant and equal to P_{max}, i.e.,

$$S_u(P_{max}) = \sqrt{\frac{P_{max}}{K}}$$

$S_{j+1}(P_{max})$ is unsteady state solution of Eq. (3-3) for P_i as constant and equal to P_{max} and $S_{j-m} = 0$. Then $S_{j+1}(P_{max})$ is given as:

$$S_{j+1}(P_{max}) = \sqrt{\frac{P_{max}}{K}} \frac{1-\exp(-2\sqrt{K P_{max}}\ m)}{1+\exp(-2\sqrt{K P_{max}}\ m)}$$

ε is a presumed tolerance equal to 0.001 cm. Then we can write:

$$S_u(P_{max}) - S_{j+1}(P_{max}) = \sqrt{\frac{P_{max}}{K}} \frac{\exp(-2\sqrt{K P_{max}}\ m)}{1+\exp(-2\sqrt{K P_{max}}\ m)} \leqq \varepsilon$$

simplifying it,

$$\frac{\exp(-2\sqrt{K P_{max}}\ m)}{1+\exp(-2\sqrt{K P_{max}}\ m)} \leqq \frac{\varepsilon}{2}\sqrt{\frac{K}{P_{max}}}$$

Let $\frac{\varepsilon}{2}\sqrt{\frac{K}{P_{max}}} = \alpha$, Then we can write:

$$m \geqq \frac{1}{2\sqrt{K P_{max}}} \quad \ell n \ \frac{1-\alpha}{\alpha} \qquad (3\text{-}8)$$

3.2 COMPUTER EXPERIMENTATION

The hypotheses of the foregoing section were incorporated in the simulation of runoff distribution based on Eqs. (3-4) and (3-5) in accordance with an assumed distribution of rainfall. The rainfall distribution was, for simplicity, assumed to be uniform:

$$f(P) = \begin{cases} \frac{1}{\theta} & 0 \leqq P \leqq \theta \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}9)$$

Note that $P_{max} = \theta$ in Eq. (3-8). In the simulation θ and K were first fixed and m was computed by Eq. (3-8). By randomly choosing P_i according to Eq. (3-9),

S_{i+1} was computed by Eq. (3-4). Computations were performed until S_j and S_{j+1} were obtained. Using Eq. (3-6), Q_j was then computed.

For the same fixed θ and K values this same procedure was repeated 1000 times and consequently 1000 random values of Q_j were obtained. Then statistical properties including mean, variance, skewness and kurtosis, of these 1000 Q_j values were calculated. This same procedure was also repeated for different values of θ and K. The simulation results are tabulated in Tables 3-1a, 3-1b, 3-2a, 3-2b, 3-3 and 3-4.

Table 3-1a shows the mean of Q_j values and Table 3-1b the ratio of the mean of Q to that of P. This ratio is plotted in Fig. 3-1 for various values of K. It is clear that the mean of Q_j values, $\bar{Q}_j$ is independent of K and θ and is equal to the mean of rainfall, μ_p. This conclusion can be seen to be true by taking the expected value of Eq. (3-1),

$$\frac{d}{dt} E[S] + E[q] = E[P]$$

Because E[S] is constant, $\frac{d}{dt} E[S]$ is zero. Then we have:

$$E[q] = E[P]$$

This leads us to conclude that the man of runoff is independent of the rainfall distribution and the parameters K and n. We can simply write:

$$\bar{Q}_j = \mu_p \qquad (3\text{-}10)$$

Table 3-2a shows variance of Q, σ_Q^2, and Table 3-2b the ratio of the variance of Q_j to that P, σ_P^2. Table 3-2b was plotted in Fig. 3-2b and eye-fit curves were drawn to the plotted points. It is clear that

Table 3-1a. Mean of output Q for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.0	99.66	91.55	81.38	68.97	60.15	55.95	49.40	40.86	30.68	20.36
	5.0	98.99	90.27	80.23	69.84	59.66	55.15	50.67	39.96	29.84	19.91
	1.0	101.40	88.32	78.51	70.23	59.40	53.98	48.66	40.02	29.52	19.51
	.5	99.93	90.48	80.43	70.20	58.81	55.29	49.80	39.92	30.38	20.18
	.1	100.08	90.46	80.42	70.73	61.18	55.34	49.95	39.88	29.51	19.97
	.05	98.35	89.39	80.40	68.16	60.73	55.29	50.53	40.13	30.45	20.35
	.01	101.08	88.12	79.98	69.84	59.27	54.55	40.54	40.54	30.12	19.96
	.005	99.57	89.90	80.92	69.40	60.19	55.69	50.53	41.09	29.88	20.06
	.001	100.36	90.31	80.44	71.35	60.27	55.16	50.14	40.11	30.02	20.07
	.0005	99.61	89.66	80.72	70.84	60.31	55.49	49.92	40.06	29.90	19.96
	.0001	100.11	89.82	79.38	70.06	60.02	55.08	50.03	39.78	30.11	20.04
P	μ	100.	90.	80.	70.	60.	55.	50.	40.	30.	20.
	σ^2	3333.33	2700.0	2133.33	1633.33	1200.0	1008.33	833.33	533.33	300.0	133.33
	μ_3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	μ_4/σ^4	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-1b. Ratio of the mean of Q to the mean of P for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.0	.997	1.017	1.017	.985	1.002	1.017	.988	1.021	1.023	1.018
	5.0	.99	1.003	1.003	.998	.994	1.003	1.013	.999	.995	.996
	1.0	1.014	.981	.981	1.003	.990	.981	.973	1.000	.984	.975
	0.5	.999	1.005	1.005	1.003	.98	1.005	.996	.998	1.013	1.009
	0.1	1.001	1.005	1.005	1.010	1.02	1.006	.999	.997	.984	.999
	0.05	.984	.993	1.005	0.974	1.012	1.005	1.011	1.003	1.015	1.017
	0.01	1.011	0.979	1.000	0.998	0.988	0.992	0.991	1.012	1.004	.998
	0.005	.996	.999	1.011	.991	1.003	1.013	1.011	1.027	.996	1.003
	0.001	1.004	1.003	1.005	1.019	1.004	1.003	1.003	1.003	1.001	1.003
	.0005	.996	.996	1.009	1.012	1.005	1.009	.998	1.001	.997	.998
	.0001	1.001	.998	.992	1.001	1.000	1.001	1.001	.994	1.004	1.002
P	μ	100.	90.	80.	70.	60.	55.0	50.	40.	30.	20.
	σ^2	3333.33	2700.00	2133.33	1633.33	1200.00	1088.33	833.33	533.33	300.00	133.33
	$\mu_3\ \sigma^3$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\mu_4\ \sigma^4$	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-2a. Variance of Q for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.	3254.10	2627.86	2071.81	1536.32	1117.87	971.73	794.58	487.40	277.23	123.96
	5.	3451.15	2589.33	2038.95	1593.47	1160.66	952.10	749.13	477.74	270.06	118.37
	1.	3006.32	2307.71	1809.47	1393.52	1040.43	832.24	710.01	459.92	248.19	101.38
	0.5	2795.22	2385.47	1866.18	1329.32	961.85	851.70	648.09	423.44	222.01	96.59
	0.1	2250.71	1832.51	1422.26	1170.72	779.91	632.29	537.56	347.74	163.91	68.10
	0.05	2157.32	1803.26	1283.20	972.11	656.17	562.90	447.11	292.98	145.90	64.35
	0.01	1585.78	1216.05	915.43	691.46	478.06	423.70	324.65	188.37	95.98	39.85
	0.005	1181.20	946.24	753.98	558.50	425.31	330.18	274.44	164.19	75.56	31.04
	0.001	781.56	618.77	459.73	335.66	229.92	185.56	154.65	86.69	42.92	15.89
	0.0005	580.48	448.20	335.84	258.26	180.74	144.67	106.19	64.20	32.01	12.66
	0.0001	317.62	234.35	176.03	122.12	84.98	74.38	56.24	33.92	15.79	6.05
P	μ	100.0	90.	80.	70.	60.	55.	50.	40.	30.	20.
	σ^2	3333.33	2700.0	2133.33	1633.33	1200.00	1008.33	833.33	533.33	300.0	133.3
	μ^3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	μ_4/σ_4	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-2b. Ratio of the variance of Q to the variance of P for various values of Q and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.	.976	.973	.971	.941	.932	.964	.953	.914	.924	.930
	5.	1.035	.959	.956	.976	.967	.944	.899	.896	.9	.888
	1.	.902	.855	.848	.853	.867	.825	.852	.862	.827	.760
	0.5	.839	.884	.875	.814	.802	.845	.778	.794	.74	.724
	0.1	.675	.679	.667	.717	.65	.627	.645	.652	.546	.511
	0.05	.647	.668	.602	.595	.547	.558	.537	.549	.486	.483
	0.01	.476	.450	.429	.423	.398	.420	.390	.353	.320	.299
	0.005	.354	.350	.353	.342	.354	.327	.329	.308	.252	.233
	0.001	.234	.229	.215	.206	.192	.184	.186	.163	.143	.119
	.0005	.174	.166	.157	.158	.151	.143	.127	.120	.107	.095
	.0001	.095	.087	.083	.075	.071	.074	.067	.064	0.053	0.045
P	μ	100.	90.	80.	70.	60.	55.0	50.	40.	30.	20.
	σ^2	3333.33	2700.00	2133.33	1633.33	1200.00	1088.33	833.33	533.33	300.00	133.33
	μ_3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	μ_4/σ^4	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-3. Skewness of Q for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.0	-0.003	-0.020	-0.020	0.033	0.037	-0.018	0.064	-0.009	-0.043	-0.050
	5.0	0.051	0.0	0.001	-0.007	0.032	0.003	0.015	0.005	0.028	0.042
	1.0	0.024	0.053	0.054	0.051	0.075	0.060	0.051	0.012	0.109	0.111
	0.5	0.029	0.020	0.023	0.072	0.048	0.031	0.113	0.059	0.039	0.091
	0.1	0.063	0.103	0.106	0.085	0.117	0.117	0.157	0.128	0.118	0.150
	0.05	0.199	0.141	0.185	0.159	0.069	0.198	0.087	0.178	0.102	0.098
	0.01	0.108	0.212	0.128	0.211	0.127	0.140	0.182	0.066	0.151	0.048
	0.005	0.156	0.097	0.152	0.179	0.169	0.173	0.249	0.207	0.231	0.108
	0.001	0.22	0.086	0.217	0.180	0.181	0.145	0.128	0.121	0.158	0.247
	0.0005	0.09	0.064	0.250	0.181	0.201	0.161	0.261	0.271	0.043	0.28
	0.0001	0.16	0.002	0.129	0.023	0.026	0.176	0.203	0.091	0.203	0.08
P	μ	100.	90.	80.	70.	60.	55.	50.	40.	30.	20.
	σ^2	3333.33	2700.0	2133.33	1633.33	1200.0	1008.33	833.33	533.33	300.0	133.3
	μ_3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	μ_4/σ^4	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-4a. Kurtosis of Q for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.	1.784	1.777	1.777	1.783	1.823	1.776	1.788	1.849	1.862	1.783
	5.	1.710	1.757	1.756	1.772	1.739	1.754	1.844	1.813	1.770	1.843
	1.	1.806	1.806	1.807	1.879	1.792	1.813	1.779	1.813	1.781	1.832
	0.5	1.805	1.716	1.763	1.842	1.795	1.771	1.796	1.803	1.838	1.823
	0.1	1.922	1.928	1.941	1.854	1.854	1.989	1.963	1.958	2.047	2.154
	0.05	1.930	1.958	1.986	1.985	2.138	2.054	2.163	2.169	2.196	2.321
	0.01	2.229	2.278	2.317	2.347	2.441	2.312	2.326	2.554	2.496	2.407
	0.005	2.438	2.494	2.429	2.487	2.437	2.469	2.446	2.497	2.796	2.622
	0.001	2.560	2.546	2.647	2.693	2.688	2.717	2.756	2.768	2.900	2.880
	0.0005	2.52	2.682	2.925	2.850	2.625	2.856	2.736	2.845	2.740	3.102
	0.0001	2.77	2.67	2.896	2.961	2.941	2.946	2.965	2.799	2.846	2.64
P	μ	100.0	90.	80.	70.	60.	55.	50.	40.	30.	20.
	σ^2	333.33	2700.0	2133.33	1633.33	1200.0	1008.33	833.33	533.33	300.0	133.33
	μ_3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	μ_4/σ^4	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

Table 3-4b. Ratio of the Kurtosis of Q to the Kurtosis of P for various values of θ and K.

	K \ θ	200	180	160	140	120	110	100	80	60	40
Q	10.	.991	.987	.987	.991	1.013	.987	.993	1.027	1.034	.991
	5.	.95	.976	.976	.984	.966	.974	1.024	1.007	.983	1.024
	1.	1.003	1.003	1.004	1.044	.996	1.007	.988	1.007	.989	1.018
	0.5	1.003	.978	.979	1.023	.997	.984	.998	1.002	1.021	1.013
	0.1	1.068	1.071	1.078	1.030	1.030	1.105	1.091	1.008	1.137	1.197
	0.05	1.072	1.088	1.103	1.103	1.171	1.141	1.202	1.205	1.220	1.289
	0.01	1.238	1.266	1.287	1.304	1.356	1.284	1.292	1.419	1.387	1.337
	0.0005	1.354	1.386	1.349	1.382	1.354	1.372	1.359	1.387	1.553	1.457
	0.001	1.422	1.414	1.471	1.496	1.493	1.509	1.531	1.538	1.611	1.600
	.0005	1.400	1.49	1.625	1.583	1.458	1.587	1.520	1.581	1.522	1.723
	.0001	1.539	1.483	1.625	1.583	1.458	1.587	1.520	1.581	1.522	1.723
P	μ	100.	90.	80.	70.	60.	55.0	50.0	40.	30.	20.
	σ^2	3333.33	2700.00	2133.33	1633.33	1200.00	1088.33	833.33	533.33	300.00	133.33
	μ_3/σ^3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\mu_4\ \sigma^4$	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80

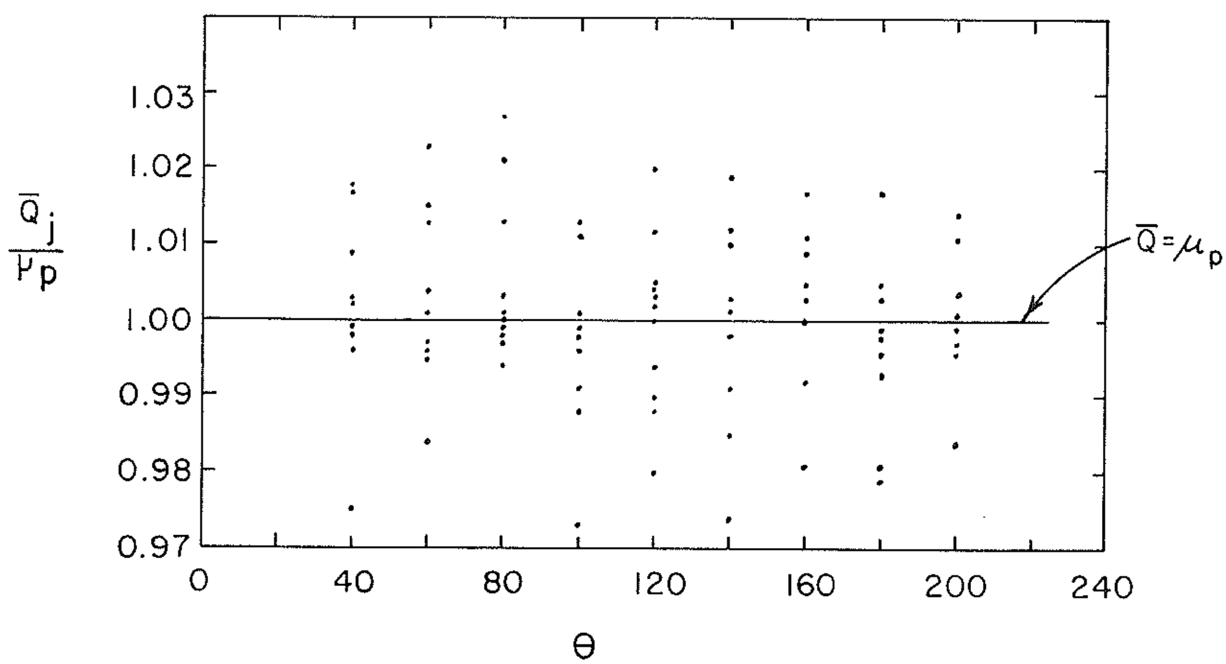

Fig. 3-1. Ratio of mean of runoff to that of rainfall for various values of K.

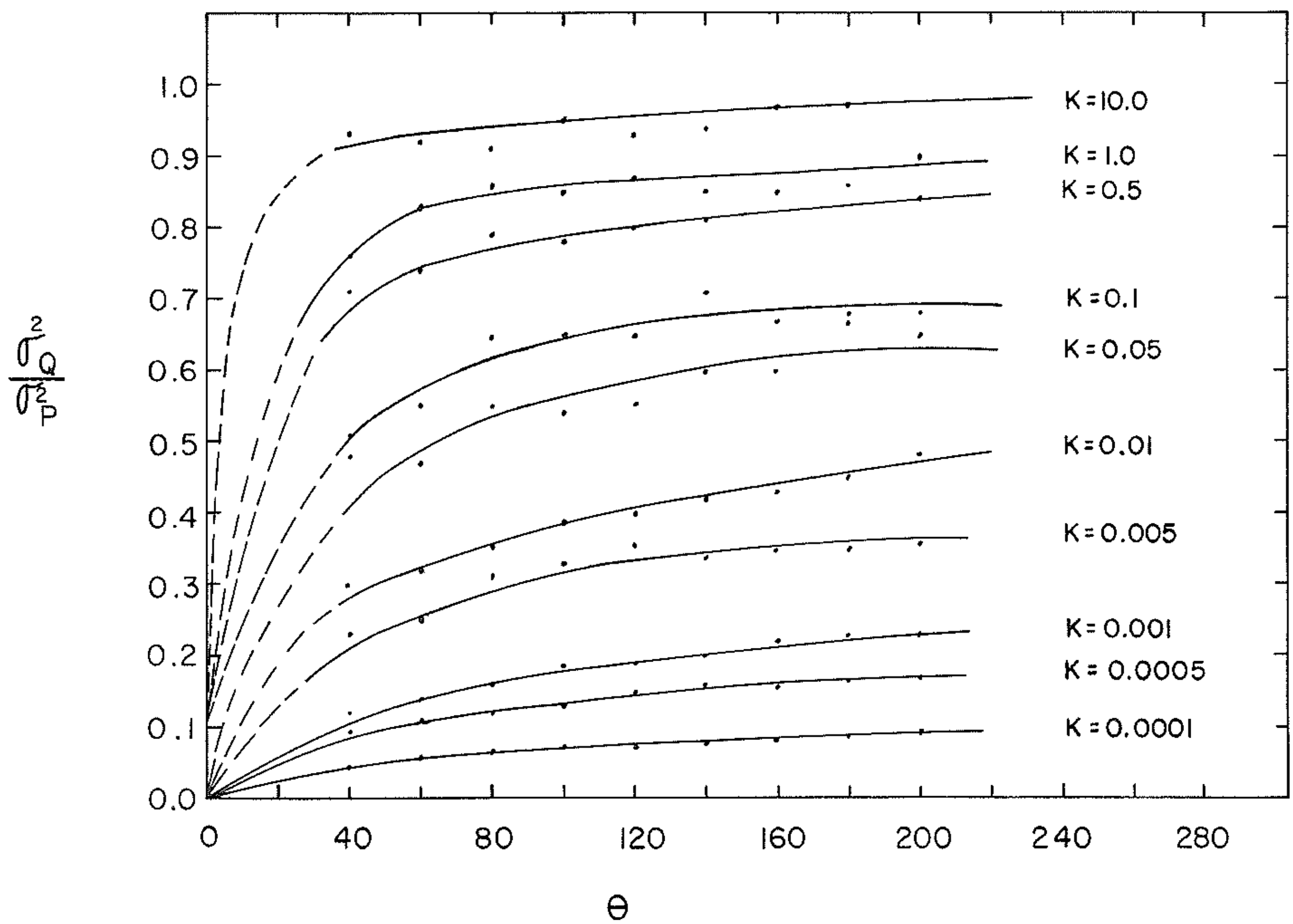

Fig. 3-2. Ratio of variance of runoff to that of rainfall for various values of K.

$$\frac{\sigma_Q^2}{\sigma_P^2} \leqq 1$$

This was seen to be true in the case of linear hypothesis.

Tables 3-3 and 3-4 show skewness and durtosis of runoff Q (see Fig. 3-3) for different K and θ. From these tables we conclude that

$$\lim_{K \to 0} \text{Skew } (Q) = 0$$

$$\lim_{K \to 0} \text{Dur } (Q) = 3$$

Hence, as K goes to zero, the runoff distribution approaches a normal distribution. To verify this conclusion the runoff distribution was simulated for K = 0.02 and θ = 110; m was 6 years. The relative frequency distribution of Q for this case is shown in Fig. 3-4. Mean, variance, skewness and kurtosis are also shown on the figure. For computations using the normal curve, $\bar{Q}$ = 55cm (from Eq. (3-9) and σ_Q^2 = 453.75 cm^2 (obtained from the family of curves by interpolation, Fig. 3-2). From Fig. 3-4 it is apparent that there is a close agreement between the simulated curve and the normal curve. Another example is shown in Fig. 3-5.

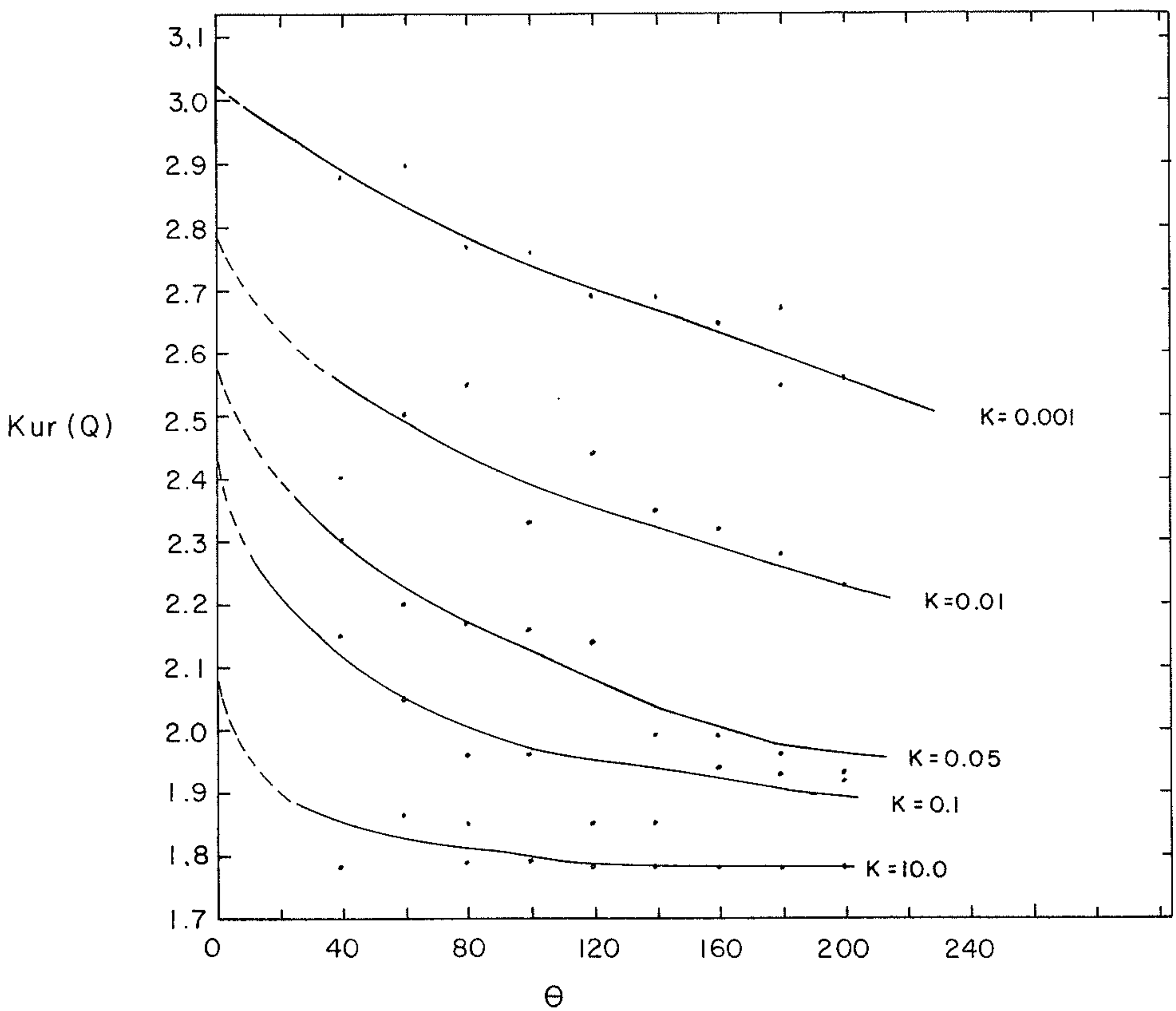

Fig. 3-3. Kurtosis of runoff for various values of K.

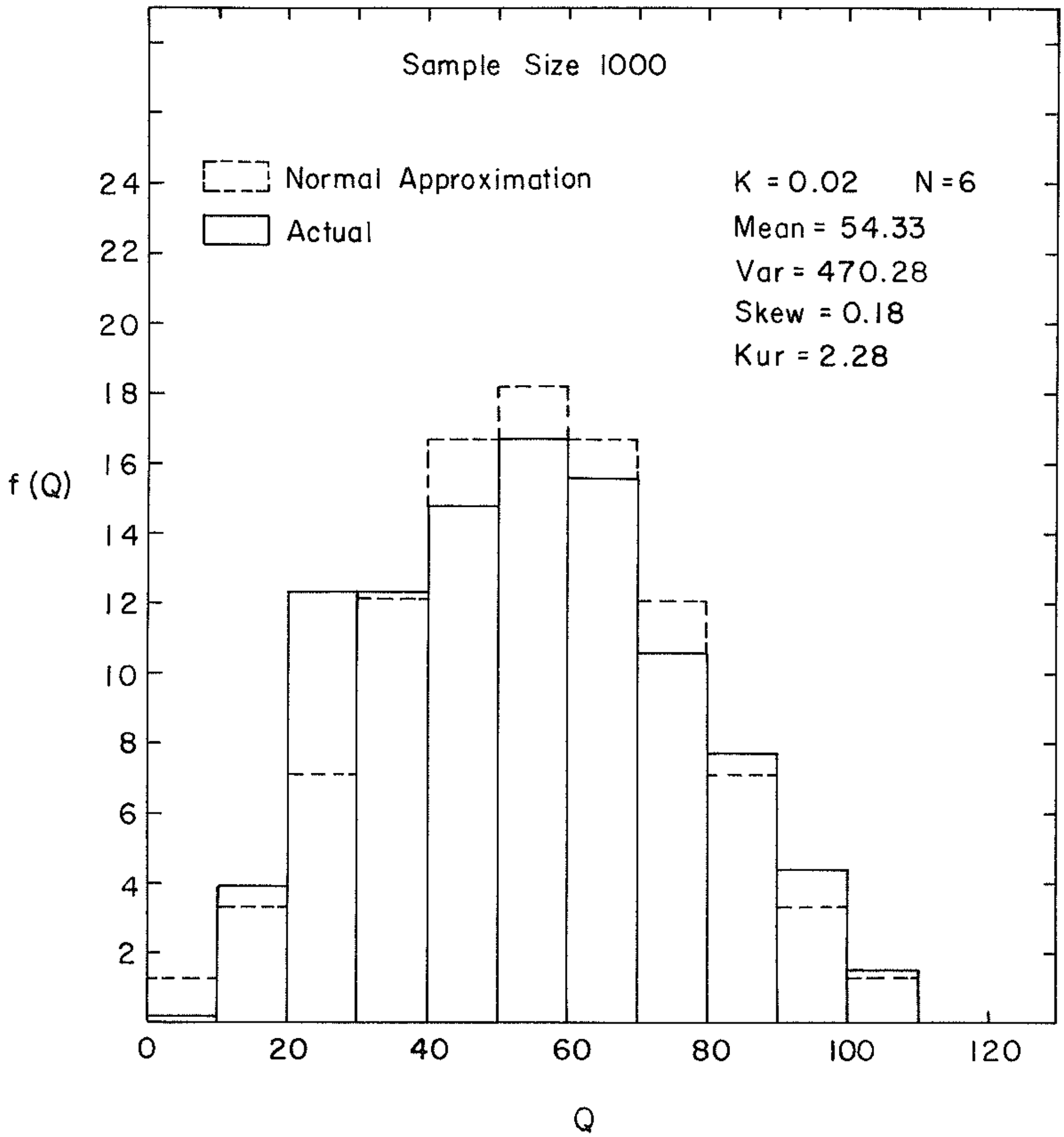

Fig. 3-4. Simulation of runoff distribution.

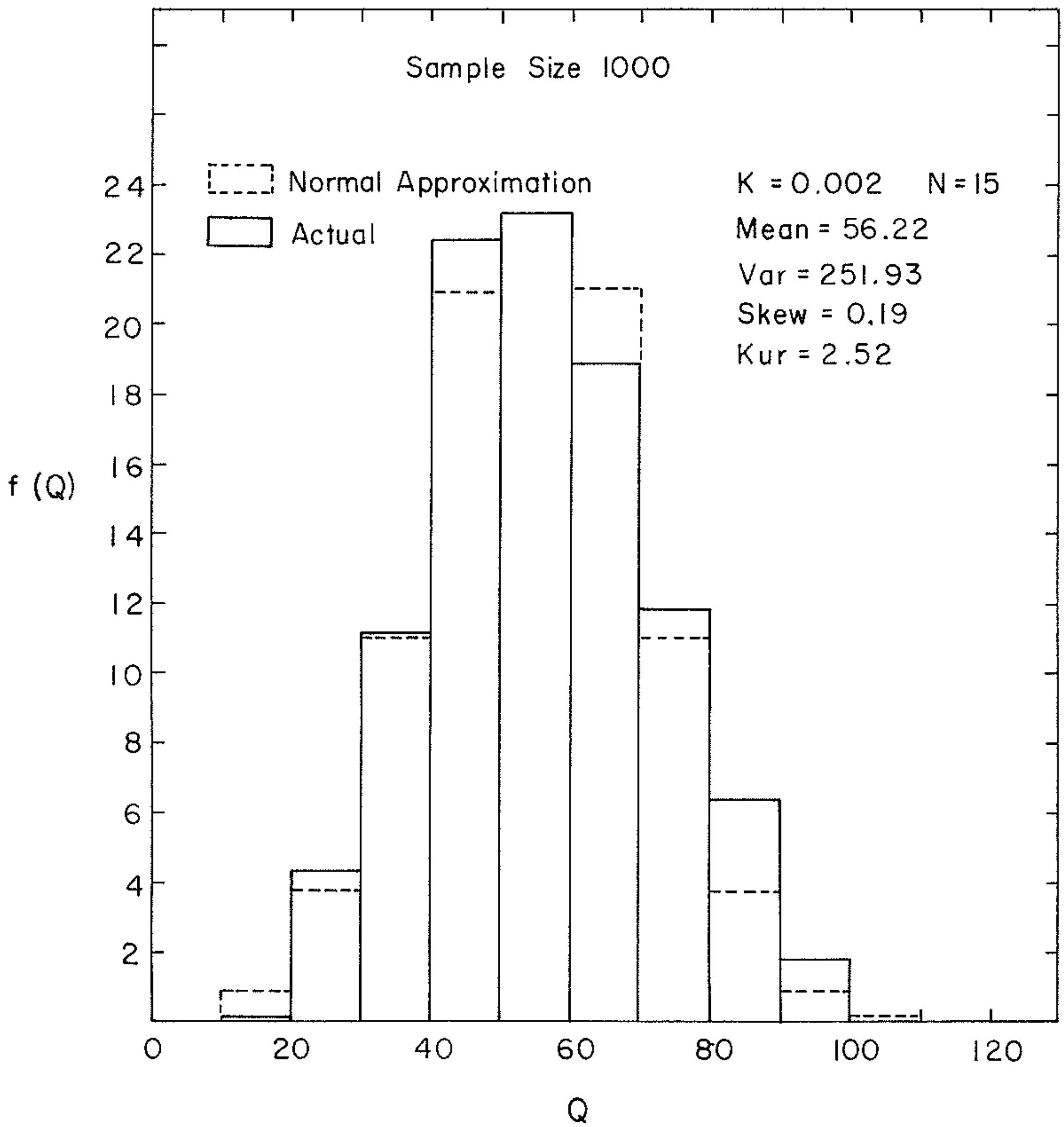

Fig. 3-5. Simulation of runoff distribution.

Chapter 4

CONCLUDING REMARKS

Under the assumption of linear transformation statistical properties of runoff have been explicitly derived in terms of those of rainfall. It has been shown both theoretically and by simulation that regardless of the distribution of rainfall, the distribution of runoff approaches a normal distribution for increasing length of carryover.

When the transformation is nonlinear an explicit derivation of the distribution parameters of runoff in terms of those of rainfall does not seem tractable. Nevertheless, the convergence of runoff distribution to a normal distribution, regardless of the distribution of rainfall, for increasing length of carryover holds.

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