June 1973 WRRI Report No. 017

SIMULATION OF COUPLED LEAKY AQUIFERS AND SURFACE - WATER SYSTEM

Technical Completion Report

Project No. A-028-NMEX

SIMULATION OF COUPLED LEAKY AQUIFERS AND SURFACE-WATER SYSTEM

Zubair A. Saleem

TECHNICAL COMPLETION REPORT

OWRR: A-028-NMEX

June 1973

New Mexico Water Resources Research Institute
in cooperation with
Department of Geoscience
New Mexico Institute of Mining and Technology

The work upon which this report is based was supported in part by funds provided through the New Mexico Water Resources Research Institute by the United States Department of Interior, Office of Water Resources Research, as authorized under the Water Resources Act of 1964, Public Law 88 - 379, under project number A-028-NMEX.

ABSTRACT

Two types of models of groundwater flow in multiaquifer systems are presented in this report. In the first part, a numerical model for simulation of groundwater flow in multiaquifer systems is developed. A system of nonlinear partial differential equations, isomorphic to groundwater flow in the multiaquifer, are formulated in a finite difference form suitable for the line successive overrelaxation technique. The resulting system of difference equations is solved using an efficient algorithm. The algorithm serves as a means for solving the bitridiagonal system of difference equations representing flow in two aquifers simultaneously during only one iteration cycle. Storativity of the semiconfining layer can be considered indirectly or can be neglected. The results derived from the numerical simulator for the flow to wells in two coupled aquifers are in good agreement with analytical results for the system.

In the second part of this report, analytical models of steady groundwater flow in a three-aquifer system are derived. The three-aquifer system consists of a valley area containing a shallow aquifer and a deep aquifer coupled through an aquitard, and a highland area containing an intake-area aquifer. Four cases of the system are analyzed: in two of the cases, the vertical variations of the hydraulic head in the shallow unconfined aquifer are assumed small; and for the other two cases, an approximate differential equation to describe the flow in the unconfined aquifer is presented. The analytical solutions were programmed for evaluation by a digital computer. The results show that a variation in the hydraulic characteristics of any one of the three aquifers affected the flow in all three aquifers and the leakage coefficient of the aquitard

plays a key role in governing the flow in the system. The use of a weighted average depth of the flow profile facilitates the derivation of analytical solutions but its use yields results that are significantly different from the case where vertical variations can be neglected.

Acknowledgements

This research was supported in part by funds provided by the New Mexico Water Resources Research Institute under OWRR project number A-028-NMEX 3109-39. I am grateful to Drs. M.S. Hantush, Chester McKee, and Donald Beaver for useful discussions. The computer time was provided by the computer centers of the University of Illinois, Chicago, and New Mexico Institute of Mining and Technology.

CONTENTS

List of Figures

Chapter 1

INTRODUCTION

Many extensive groundwater systems comprise several aquifers hydraulically connected to one another through layers which are usually semiconfining. The aquifers respond conjunctively to stresses imposed on the system. Such coupling of aquifers necessitates, for the analysis of the flow of groundwater through them, the simultaneous consideration of the entire system. The results of analysis of groundwater flow through aquifers determines their excitation/response which is required for the management of the system [Saleem and Jacob, 1971; Young and Bredehoeft, 1972].

Geohydrologists and petroleum engineers have traditionally employed analytical techniques to solve problems of flow of fluids through porous media. Analytical solutions for subsurface flow of fluids have been presented by several investigators [e.g., Theis, 1935; Jacob, 1940; Muskat, 1946; Hantush, 1964; Freeze and Witherspoon, 1966]. Transient flow to wells in leaky aquifers was analyzed by Hantush and Jacob [1955] and Hantush [1960]. Hantush [1967] also derived solutions for drawdowns due to wells in coupled leaky aquifers. Several other investigators have presented analytical solutions to problems of steady and unsteady flow in two-layered systems [e.g., Hantush and Jacob, 1955; Jacquard, 1960; Lefkovitz, et al., 1961; Papadopulos, 1966; Freeze and Witherspoon, 1966; Saleem and Jacob, 1969; Khan et al., 1971; and Toksöz and Kirkham, 1971]. Neumann and Witherspoon [1969, 1971] presented more complete solutions for drawdowns due to wells in two- and three-layered aquifer systems and Javandel and Witherspoon [1969] verified some of their solutions.

Analytical solutions of groundwater problems are useful, in that they provide insight for understanding and evaluating the behavior of groundwater systems. They are usually available for and applied to situations which can be represented closely by idealized homogeneous and isotropic porous media. However, actual groundwater systems are rarely homogeneous and isotropic and therefore are often simplified to conform to problems represented by analytical solutions, Furthermore, analytical solutions are sometimes difficult to derive for complex systems or are so involved that they require excessive computer time for evaluation. Because of the above inadequacies in analytical techniques several investigators have become interested in techniques of numerical analysis.

Two types of models are developed in this report. The first is a digital computer model simulating groundwater flow in multiaquifer systems. The aquifers can be isotropic or anisotropic with boundary conditions of the first kind (Dirichlet type) or of the second kind (Neumann type). The second are analytical models of one-dimensional flow in systems comprising three interacting aquifers which represent idealized textbook type artesian basins. The analytical solutions are programmed on a digital computer for evaluation and some typical results, using parameters selected from the Roswell Basin, New Mexico, are presented.

References

Freeze, R. Allan, and P.A. Witherspoon, Theoretical analysis of regional groundwater flow: 1. Analytical and numerical solutions to the mathematical model, Water Resour. Res., 2(4), 641-656, 1966.

Hantush, M. S., Modification of the theory of leaky aquifers, J. Geophys. Res., 65(11), 1960.

Hantush, M.S., Hydraulics of wells, in Advances in Hydroscience, V.T. Chow, ed., v. 1, 281-442, Acad. Press, New York, 1964.

Hantush, M.S., Flow to wells in aquifers separated by a semi-pervious layer, J. Geophys. Res., 72 16, 1709-1720, 1967.

Hantush, M.S., and C.E. Jacob, Nonsteady radial flow in an infinite leaky aquifer, Trans. Am. Geophys. Union, 36(1), 95-100, 1955.

Hantush, M.S., and C.E. Jacob, Steady three-dimensional flow to a well in a two-layered aquifer, Trans. Am. Geophys. Union, 36(2), 286-292, 1955.

Jacob, C.E., On the flow of water in an elastic artesian aquifer, Trans. Am. Geophys. Union, 21, 574-586, 1940.

Jacquard, P., Étude mathematique du drainage d'un réservoir hétérogène, Rev. Inst. Francais du Pétrole, XV, 10, 1384, 1960.

Javandel, I., and P.A. Witherspoon, A method of analyzing transient fluid flow in multilayered aquifers, Water Resour. Res., 5(4), 856-869, 1969.

Khan, M.Y., Don Kirkham, and S. Töksoz, Steady state flow around a well in an two-layered aquifer, Water Resour. Res., 7(1), 155-165, 1971.

Lefkovitz, H.C., P. Hazelbrock, E.E. Allen, and C.S. Mathews, A study of the behavior of bounded reservoirs composed of stratified layers, Jour. Soc. Petrol. Engrs., 1, 43-58, 1961.

Muskat, M., The Flow of Homogeneous Fluids Through Porous Media, J.W. Edwards, Inc., Ann Arbor, Michigan, 1946.

Neumann, S.P., and P.A. Witherspoon, Theory of flow in a confined two aquifer system, Water Resour. Res., 5(4), 803-816, 1969.

Neumann, S.P., and P.A. Witherspoon, Flow in multiple aquifer systems, in Sea-Water Intrusion: Aquitards in the Coastal Groundwater Basin of Oxnard Plain, Ventura County, California Dept. of Water Resources Bull. No. 63-4, 21-36, 1971.

Papadopulos, S.S., Nonsteady flow to multiaquifer wells, J. Geophys. Res., 71(20), 4791-4797, 1966.

Saleem, Z.A., and C.E. Jacob, Drawdown distribution due to well fields in coupled leaky aquifer systems, abs. in E.O.S. (Trans. Am. Geophys. Union), 50(11), p. 616, 1969.

Saleem, Z.A., and C.E. Jacob, Optimal use of coupled leaky aquifers, Water Resour. Res., 7(2), 382-393, 1971.

Theis, C.V., The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage, Trans. Am. Geophys. Union, 16, 519-524, 1935.

Töksoz, S., and Don Kirkham, Steady drainage of soils: I, theory, Proc. Amer. Amer. Soc. Civil Eng., 97(IR1), 1-18, 1971.

Young, R.A., and J.D. Bredehoeft, Digital simulation for solving management problems of conjunctive groundwater and surface water systems, 8(3), 533-556, 1972.

Chapter 2

NUMERICAL MODEL

Introduction

Simulation of groundwater flow on digital computers in single aquifers has been studied by several investigators [for a summary of previous works, see Pinder and Bredehoeft, 1968; Freeze, 1971]. Remson et al. [1971] give an outline of selected published applications of most of the numerical techniques used in problems of flow through porous media. Freeze [1971] recently presented a three-dimensional model of transient saturated-unsaturated groundwater flow in aquifers. Javandel and Witherspoon [1969] and Neumann and Witherspoon [1971] used the finite element method for numerical simulation of flow to wells in multiple aquifers.

Bredehoeft and Pinder [1970] presented a simulation model for multiple aquifers suitable for the study of excitation/response on a basin-wide basis. They used the iterative alterating direction implicit procedure (ADIPIT) for numerical simulation and considered storativity of the aquitard indirectly in the model. The solution for the distribution of head in each of the two aquifers of the system is derived by iteration for the first aquifer, for the second aquifer, and then repeating the process till convergence is obtained. Prickett and Lonnquist [1971] used a similar procedure for simulation of groundwater flow in multiaquifers.

In this study the nonlinear partial differential equations representing groundwater flow in interacting aquifers are written in a finite difference form suitable for the line successive overrelaxation (LSOR) technique. The resulting system of difference equations is solved simultaneously using a more efficient procedure than has been presented before.

Mathematical Formulation

Groundwater flow in a two aquifer system (Fig. 2-1) is governed by equations (1) and (3) below. Equation (1) is the approximate differential equation for describing the flow in the water-table aquifer and equation (3) is the equation of flow for the confined aquifer. It is assumed that the principal components of the transmissivity and hydraulic conductivity tensors are parallel with the coordinate axes, that contrasts between the hydraulic conductivities of the two aquifers and the semiconfining layer (aquitard) are so great that the flow is essentially vertical in the semiconfining layer and horizontal in the aquifers, and that the storativity in the semipervious layer is negligible.

$$\frac{\partial}{\partial x}\left(K_{xx} b \frac{\partial D}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy} b \frac{\partial D}{\partial y}\right) = S_1 \frac{\partial D}{\partial t} + W_1(x,y,t) \qquad (1)$$

where

$$W_1(x,y,t) = q_1(x,y,t) - W_s(x,y,t) - W_c(x,y,t) - W_p(x,y,t) - L(x,y,t), \qquad (2)$$

$q_1(x,y,t)$ — volumetric rate of groundwater withdrawal per unit area;

$W_s(x,y,t)$, $W_c(x,y,t)$, $W_p(x,y,t)$ — volumetric rates, per unit area, of replenishment of the aquifer from return flows from two different sources of water and from precipitation, respectively;

K_{xx}, K_{yy} — principal components of the hydraulic conductivity tensor, assuming the principal components are collinear with the coordinate axes;

$b(x,y,t)$ — saturated thickness of the water-table aquifer;

$D(x,y,t)$ hydraulic head in the water-table aquifer;

$L(x,y,t)$ volumetric rate of leakage per unit area;

$S_1(x,y)$ storativity of the water-table aquifer.

$$\frac{\partial}{\partial x}(T_{xx}\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(T_{yy}\frac{\partial h}{\partial y}) = S_2\frac{\partial h}{\partial t}) = W_2(x,y,t), \qquad (3)$$

where

$$W_2(x,y,t) = q_2(x,y,t) + L(x,y,t) \qquad (4)$$

and

$$L = \frac{K'}{b'}(h-D). \qquad (5)$$

K'/b' leakance or leakage coefficient; $\underline{K'}$ and $\underline{b'}$ are the hydraulic conductivity and the thickness of the aquitard, respectively;

$h(x,y,t)$ hydraulic head in the confined aquifer;

$S_2(x,y)$ storativity of the confined aquifer;

T_{xx}, T_{yy} principal components of the transmissivity tensor;

$q_2(x,y,t)$ volumetric rate of groundwater withdrawal per unit area.

Numerical Solution

Equations (1) and (3) are coupled nonlinear partial differential equations. The principal components of the hydraulic conductivity tensor and transmissivity tensor which occur in these equations are functions of the space variables. The saturated thickness of the water-table aquifer, b, is a function of the dependent variable D. Therefore, an iterative numerical scheme using implicit finite difference formulations is preferred for solving the system of equations because they are convergent, unconditionally stable, and allow time steps to be selected independent of space increments.

The line successive overrelaxation (LSOR) technique is employed in this study. The LSOR technique [Young, 1962] is an iterative solution technique and was successfully used by Cooley [1971] and Freeze [1971] to simulate saturated-unsaturated flow problems. Cooley compared ADIPIT with LSOR and found the latter to be faster and easier to use for the problem he studied.

The nonlinear partial differential equation (1) is written in a finite difference form using the backward difference method:

$$K_{xxi+\frac{1}{2},j}(D-E)^{n+1}_{i+\frac{1}{2},j}\frac{D^{n+1}_{i+1,j}-D^{n+1}_{i,j}}{\Delta x_i \Delta x_{i+\frac{1}{2}}}$$

$$-K_{xxi-\frac{1}{2},j}(D-E)^{n+1}_{i-\frac{1}{2},j}\frac{D^{n+1}_{i,j}-D^{n+1}_{i-1,j}}{\Delta x_i \Delta x_{i-\frac{1}{2}}}$$

$$+K_{yyi,j+\frac{1}{2}}(D-E)^{n+1}_{i,j+\frac{1}{2}}\frac{D^{n+1}_{i,j+1}-D^{n+1}_{i,j}}{\Delta y_j \Delta y_{j+\frac{1}{2}}}$$

$$-K_{yyi,j-\frac{1}{2}}(D-E)^{n+1}_{i,j-\frac{1}{2}}\frac{D^{n+1}_{i,j}-D^{n+1}_{i,j-1}}{\Delta y_j \Delta y_{j-\frac{1}{2}}}$$

$$+\frac{K'_{i,j}}{b'_{i,j}}(h^{n+1}_{i,j} - D^{n+1}_{i,j}) = \frac{S_{1_{i,j}}}{\Delta t^{n+\frac{1}{2}}}(D^{n+1}_{i,j} - D^{n}_{i,j})$$

$$+q_{1_{i,j}}^{n} - W^{n}_{s_{i,j}} - W^{n}_{c_{i,j}} - W^{n}_{p_{i,j}}. \qquad (6)$$

Where E is elevation of the bottom of the water-table aquifer.

Similarly, equation (3) is written in a finite difference form as:

$$T_{xxi+\frac{1}{2},j}\frac{h^{n+1}_{i+1,j} - h^{n+1}_{i,j}}{\Delta x_i \Delta x_{i+\frac{1}{2}}} - T_{xxi-\frac{1}{2},j}\frac{h^{n+1}_{i,j} - h^{n+1}_{i-1,j}}{\Delta x_i \Delta x_{i-\frac{1}{2}}}$$

$$+T_{yyi,j+\frac{1}{2}}\frac{h^{n+1}_{i,j+1} - h^{n+1}_{i,j}}{\Delta y_j \Delta y_{j+\frac{1}{2}}} - T_{yyi,j-\frac{1}{2}}\frac{h^{n+1}_{i,j} - h^{n+1}_{i,j-1}}{\Delta y_j \Delta y_{j-\frac{1}{2}}}$$

$$-\frac{K'_{i,j}}{b'_{i,j}}(h^{n+1}_{i,j} - D^{n+1}_{i,j}) = \frac{S_{2_{i,j}}}{\Delta t^{n+\frac{1}{2}}}(h^{n+1}_{i,j} - h^{n}_{i,j}) + q^{n}_{2_{i,j}}, \qquad (7)$$

where $\underline{i}$ and $\underline{j}$ are indices in the $\underline{x}$ and $\underline{y}$ dimensions, respectively; Δx_i and Δy_j are incremental distances in the $\underline{x}$ and $\underline{y}$ directions, respectively; and $\underline{n}$ is the index in time. The finite difference formulations are such that mesh spacings can be varied and aquifers can be inhomogeneous and anisotropic provided the principal axes are lined up with the coordinate axes.

The finite difference equations for LSOR technique are arranged such that during the scanning along a column, values of head on adjacent columns as calculated from the most recent iteration, and all values of head at the previous time $\underline{n}$ are considered to be known values. After regrouping equations (6) and (7) in this manner, they are written for an interior node (subscript $\underline{i},\underline{j}$) as:

$$A^k_{i,j} D^{k+1}_{i,j-1} + B^k_{i,j} D^{k+1}_{i,j} + C_{i,j} h^{k+1}_{i,j} + E^k_{i,j} D^{k+1}_{i,j+1} = F^k_{i,j} \ , \qquad 8(a)$$

$$G_{i,j} h^{k+1}_{i,j-1} + H_{i,j} h^{k+1}_{i,j} + C_{i,j} D^{k+1}_{i,j} + M_{i,j} h^{k+1}_{i,j+1} = N^k_{i,j} \ ; \qquad 8(b)$$

and $$\hat{D}^{k+1}_{i,j} = \omega \, [D^{k+1}_{i,j} - D^k_{i,j}] + D^k_{i,j} \ , \qquad 9(a)$$

$$\hat{h}^{k+1}_{i,j} = \omega \, [h^{k+1}_{i,j} - h^k_{i,j}] + h^k_{i,j} \ , \qquad 9(b)$$

$$0 < \omega < 2 \ , \qquad 9(c)$$

where superscripts k and k+1 refer to the previous iterate and the most recent iterate, respectively. $D^{k+1}_{i,j}$ and $h^{k+1}_{i,j}$ are the values of heads obtained by the solution of equation (8) and $\hat{D}^{k+1}_{i,j}$ and $\hat{h}^{k+1}_{i,j}$ are the extrapolated values of heads. $A^k_{i,j}$, $B^k_{i,j}$, $C_{i,j}$, $E^k_{i,j}$, $F^k_{i,j}$, $G_{i,j}$, $H_{i,j}$, $M_{i,j}$, and $N^k_{i,j}$ are coefficients formed from the coefficients in equations (6) and (7). Coefficients with superscripts are time dependent and are evaluated at the beginning of each iteration, whereas coefficients without superscripts are evaluated in the beginning of a problem and do not change as solution progresses with time.

The set of equations (8) form a bitridiagonal system of equations and are such that they cannot be solved by the well-known triangular decomposition technique used in both ADIPIT and LSOR procedures. However, Douglas et al. [1959] developed an efficient algorithm for a system of equations which are tridiagonal in two dependent variables. This effective algorithm was employed also by Heron and von Rosenberg [1966] to solve the equations describing the transient behavior of a counter-current heat exchanger. Equations (8) are applied to the boundary points along the eight boundaries of the two aquifers and the resultant equations are solved simultaneously, using the bitridiagonal algorithm for all h and D along a line using the most recent values of these two dependent variables on adjacent lines.

After $\underline{h}$ and $\underline{D}$ are calculated, convergence at that time step is checked. If convergence is not reached to a desired tolerance, the calculated values of $\underline{h}$ and $\underline{D}$ are relaxed. time dependent coefficients are recalculated using the most recent values of the dependent variables, and the set of equations is solved again using the latest values of coefficients. The steps are repeated until convergence is obtained.

For the first iteration of the first time step, the initial values of the head are employed to estimate coefficients whereas for the first iterations of subsequent time steps, values of the head are predicted using (10). Equations (10) are derived based on the assumption that the time rates of change of heads at two consecutive time steps are approximately equal.

$$h_{i,j}^{n+1} = h_{i,j}^{n} + \lambda R^{n}(h_{i,j}^{n} - h_{i,j}^{n-1}) \tag{10a}$$

$$D_{i,j}^{n+1} = D_{i,j}^{n} + \lambda R^{n}(D_{i,j}^{n} - D_{i,j}^{n-1}) \ , \tag{10b}$$

where

$$R^{n} = \frac{\Delta t^{n+\frac{1}{2}}}{\Delta t^{n-\frac{1}{2}}} \ ; \tag{11}$$

λ is a weighting factor; and $h_{i,j}^{n+1}$ and $D_{i,j}^{n+1}$ are the extrapolated values of heads for the (n+1)th time step. Freeze [1971] used a value of 1/2 for λ, but in this study it was found that convergence resulted faster when λ is varied as the solution progresses in time. Initially λ was set equal to 1/2 and then gradually increased to 1 for large $\underline{n}$. Application of equations (10) proved to be quite effective for deriving extrapolated values of heads in the two aquifers. Cooley [1971] and Freeze [1971] discuss in more detail the LSOR technique as used by them.

The equations for flow of groundwater in interacting aquifers were solved in terms of head changes (drawdowns) with respect to a datum piezometric or groundwater surface because higher accuracy is achieved in this manner. The procedure of solving for change in head instead of head is similar to that of working with residuals except that it is relatively easier to use. McCracken and Dorn [1964] describe the approach of working with residuals. Breitenbach et al. [1968] and Cooley [1971], among others, used the technique of residuals.

Equations (6) and (7) are the finite difference forms of the basic flow equations which are based on the principle of conservation of mass. Therefore, in the solution of (6) and (7), the total amount of water in the aquifer system should be conserved. For this reason, the material balance check provides an independent measure of the accuracy with which equations (6) and (7) are being solved. A running check of the material balance is made by simply computing how much water goes in each of the aquifers, how much comes out, and how much change in storage occurred during each time step. The relative error was generally less than 1 percent but was about 3 percent for extremely large time steps.

Some of the coefficients in equations (8) include the transmissivities and hydraulic conductivities at the internodal boundaries, that is, values at ($i\pm\frac{1}{2}$) and at ($j\pm\frac{1}{2}$) positions. These internodal properties are evaluated, using the series flow law, in terms of the values at the two modes adjacent to the internodal boundaries. This procedure results in better representation of the actual flow conditions and is, therefore, more accurate than replacing the internodal values by the arithmetic average of the values of the two adjacent nodes. However, it slightly increases the number of computations needed for the calculation of coefficients in equations (8).

The program is written so that the size of space increments in the $\underline{x}$ and $\underline{y}$ directions can be varied. This feature is desirable because in developed areas of a basin small increments can be chosen to insure high accuracy, compared with other areas of less interest where detailed simulation is not needed. Because the finite difference formulation is implicit, space increments of any arbitrary size can be selected. A flow chart for the computer program is given in (Fig. 2-2).

Two kinds of boundary conditions can be simulated with the program; i.e., the constant head type (Dirichlet condition) and the constant discharge type (Neumann condition). Boundary conditions can be time dependent, however, in the problems analyzed with the program they were kept invariant with respect to time. Sources and sinks, representing recharge and pumping areas, respectively, can also be simulated.

The time step size is increased as the solution progresses in time and the change in the time step size is determined so that certain selected criteria are satisfied. In the program, the time step size is determined at the beginning of each time step, based on one of the following criteria: (1) a constant rate of increase from the previous time step; (2) change based on the percentage change of heads during the last two time steps; and (3) a change in the step size based on the number of iterations performed to obtain a convergence of the solution to a desired tolerance limit during the previous step. The time step size to be used for the next step is the one which is the least of all the three time steps.

The model can be used to simulate groundwater flow in an arbitrary number of aquifers. However, the required computer storage and the computation time needed for obtaining the solution will increase somewhat in direct proportion to the number of aquifers. Since this program yields the solution for two aquifers at a time, it is more elegant than those in which computations are made for one aquifer at a time. For example: to obtain the solution for flow in a three-aquifer system, one cycle of iteration using the ADIPIT technique consists of making computations for the first aquifer followed by the second aquifer and then for the third aquifer; whereas one cycle of iteration with the program developed here comprises only two steps: (1) making computations for the first two aquifers simultaneously, and (2) making computations for the second and third aquifers simultaneously. However, the time required for one cycle of iteration with this model may be different from the corresponding time needed using ADIPIT.

Several investigators have pointed out that the storativity of semi-pervious layers is significant and should be considered in the analysis of groundwater flow in systems including such layers [e.g., Hantush, 1960; Domenico and Mifflin, 1965; Neumann and Witherspoon, 1969, 1971]. A more rigorous simulation of interacting aquifers requires that the flow be considered in three dimensions. However, this would entail excessively large computer storage and increase computation time, thus discouraging simulation of actual basins.

Comparison with Analytical Solutions

Analytical solutions of simple groundwater problems are useful for checking the accuracy of the numerical simulation models. The numerical models are verified in this manner and used to analyze complex flow problems because rigorous proofs of the stability and convergence of the numerical simulation models of these problems often are not available. A few problems are solved here using the numerical simulator and the performance of the model in solving these problems is considered a major test of its validity.

Wells in Isotropic Aquifer Systems

The first case considered is of wells in an aquifer system comprising two infinite aquifers separated by an aquitard of negligible storage. Hantush [1967] derived a solution to the general problem of wells in two aquifers separated by an aquitard of negligible storage. He also presented a solution of the problem, which is a special case of the general solution, in which the two aquifers have identical hydraulic diffusivities. The special case is modified further for the case when the two aquifers have identical transmissivity and storativity values:

$$s_n(r,t) = \frac{Q}{8\pi T} [W(u) - W(u,\beta)], \quad (12a)$$

$$s_p(r,t) = \frac{Q}{8\pi T} [W(u) + W(u,\beta)], \quad (12b)$$

where

Q = the discharge of a well steadily pumping from the aquifer;

s_n, s_p = induced drawdown in the nonpumped aquifer and the drawdown in the pumped aquifer, respectively;

$u = r^2/(4\nu t)$; r is the distance from the center of the well, $\nu(=T/S)$ is the hydraulic diffusivity, and t is the time since pumping started;

$\beta^2 = 2r^2/B^2$, Where B is the leakage factor and is equal to $\sqrt{T/(K'/b')}$.

$W(u)$, $W(u,\beta)$ = well functions for nonleaky and leaky aquifers.

The results obtained using the numerical model are compared with the analytical solutions of the same problems derived using equations (12) and are shown in Figures 2-3, and 2-4 . Equations (12) are valid for wells in an infinite, homogeneous, and isotropic aquifer system shown as an insert in Figure 2-3 . For the numerical solution, the well was placed in the center of the grid map, which is of finite size, and the grid size was increased with distance from the well. This procedure of locating the well in the center of the grid map was found to be unsuitable because the influence of pumpage reached the outermost nodes in a short period of time thereby violating the assumption of the infinite extent of the aquifers, one of the assumptions underlying the derivation of equation (12). Any further increase in the size of space increments reduced the accuracy of the numerical results. The problem was partly overcome by locating the well in the cornermost node of the grid, reducing the strength of the well by one quarter, and then gradually increasing the node spacing away from that corner.

Figure 2-3 shows the numerical results compared with analytical solutions for the case when the lower aquifer is being pumped. The upper two curves are for the drawdown in the pumped aquifer and the lower two curves are for the induced drawdown in the upper aquifer. The agreement between the numerical and the analytical results is good except for dimensionless time greater than 200, when the reflection from the finite boundaries of the simulated aquifer influenced the nodes corresponding to the curve $r/B = 1{\cdot}0$.

In Figure 2-4, numerical and analytical results are shown for the case when the well taps the upper unconfined aquifer. The size of the space increments and the location of the well are the same as in the previous case. In addition to the boundary effects, as in the previous case, the effect of the decrease in the saturated thickness of an unconfined aquifer as water is pumped from it is present. The agreement between the numerical results and the analytical solutions was good except for large drawdowns. This discrepancy arose because the numerical results are for the case when the upper aquifer is unconfined and the analytical solutions (Equations 12) are valid for a confined aquifer system. The anomaly between the two solutions disappears when a correction for the decrease in the saturated thickness of the upper aquifer is applied. The correction is equal to $s^2/(2m)$, where s is the drawdown and m is the initial saturated thickness of the water-table aquifer; this correction is discussed by Jacob [1963]. The discrepancy between the induced drawdowns for the lower aquifer — confined in both solutions -- results because the two aquifers are coupled. The decrease in the saturated thickness of the unconfined aquifer results in a drawdown which is larger than in the case of a confined aquifer, thus changing the rate of leakage reflected by the deviation between the two results (analytical and numerical) for the induced drawdowns in the confined aquifer.

Wells in Anisotropic Aquifer Systems

Simulation of groundwater flow in coupled anisotropic aquifers using the digital computer model is discussed here, using the same set of space increments and other data as in the case of the isotropic aquifer system, except for the transmissivity and hydraulic conductivity values. Five aquifer systems (Fig. 2-5) were simulated using five different values for the anisotropic ratio, $\underline{\alpha}$ $(=T_{xx}/T_{yy}=K_{xx}/K_{yy})$; in each case, $\underline{\alpha}$ was kept the same for the two aquifers; all five aquifer systems have the same $\underline{Tg}$ $(=\sqrt{Txx\ Tyy})$ values.

Figure 2-5 shows that as the anisotropy of an aquifer system increases, the drawdonws at any given time due to a well in one of the two aquifers of the system decrease; they are maximum in the isotropic system (α=1) at the same time. This can be seen also in Figure 2-6, where dimensionless drawdown is plotted against $\underline{\alpha}$ with dimensionless time $\underline{t_d}$ $[=tTg/(r^2S)]$ as the parameter of the curves. Two drawdown curves are shown in Figure 2-5 for each value of $\underline{t_d}$; the upper ones are drawdowns in the pumped aquifer and the lower ones are induced drawdowns in the other aquifer. As in the isotropic system, the drawdowns in the two aquifers of the system tend to equalize with time and the rate of convergence to a common value of the drawdown increases with an increase in $\underline{\alpha}$. The decrease in drawdowns and increase in the rate of convergence of the drawdown curves with the increase in anisotropy shows that the anisotropic aquifers have an effective transmissivity different from $\underline{Tg}$ or they may have no single value for effective transmissivity. In general, in a coupled aquifer system, the rate of convergence of the drawdown curves with time increases with an increase in the leakage coefficient of the aquitard, with an increase in the distance of an observation point from the well, and with a decrease of storativities of the aquifers.

Summary and Conclusions

The numerical simulation models of groundwater flow systems should not require excessive computer storage and should yield results in a reasonable amount of computer time. The numerical simulation model developed here is based on the LSOR technique and uses an efficient algorithm for the simultaneous solution of the bitridiagonal system of difference equations. The model is set up for two aquifers separated by an aquitard and the storage in the aquitard can be considered indirectly.

The numerical simulation model was used to simulate groundwater flow in isotropic and in anisotropic coupled aquifer systems. Drawdowns in the two aquifers of the system, due to a well pumping from one of the two aquifers, tend to equalize: as the pumpage increases; as the leakage coefficient of the aquitard increases; as the distance of the observation point from the well increases; and as the hydraulic diffusivities of the aquifers increase. The curves indicate that the effective transmissivity of an anisotropic aquifer system is different from that given by the geometric mean of the two transmissivity coefficients. In the case where the upper unconfined aquifer is being pumped, Jacob's correction for analyzing drawdown data from an unconfined aquifer, using the theory of flow in confined aquifers, was found to be satisfactory for practical purposes.

The model is set up for two interacting aquifers but can be used for multiaquifer systems. The storage in the aquitard can be considered indirectly using the procedure suggested by Bredehoeft and Pinder (1971).

References

Bredehoeft, J.D., and G.F. Pinder, Digital analysis of areal flow in multiaquifer groundwater systems: A quasi three-dimensional model, Water Resour. Res., 6(3), 883-888, 1970.

Breitenbach, E.A., D.G. Thurnau, and H.K. van Poolen, Solution of the immiscible fluid flow simulation equations, Pap.

SPE #2021, 13pp. Soc. of Petrol. Engrs., A.I.M.E., Dallas, Texas, 1968.

Cooley, R.L., A finite difference method for unsteady flow in variably saturated porous media: Application to a single pumping well, Water Resour. Res., 7(6), 1607-1625, 1971.

Domenico, P.A., and M.D. Mifflin, Water from low-permeability sediments and land subsidence, Water Resour. Res., 1(4), 563-676, 1965.

Douglas, J., D.W. Peaceman and H.H. Rachford, A method for calculating multidimensional immiscible displacement, Petroleum Trans., A.I.M.E., v. 216, 297-306, 1959.

Freeze, R. Allan, Three-dimensional, transcient, saturated-unsaturated flow in a groundwater basin, Water Resour. Res., 7(2), 347-366, 1971.

Hantush,M.S., Modification of the theory of leaky aquifers, J. Geophys. Res., 65(11), 1960.

Hantush, M.S., Flow to wells in aquifers separated by a semipervious layer, J. Geophys. Res., 72(6), 1709-1720, 1967.

Heron, E.H., and D.V. von Rosenberg, An efficient numerical method for the solution of pure convective transport problems with split boundary conditions, Chem, Eng. Sci., 21, 337-342, 1966.

Jacob, C.E., Determining the permeability of water-table aquifers, in: Methods of Determining Permeability, Transmissibility and Drawdown, U.S. Geol. Survey Water-Supply Paper 1536-I, 245-271, 1963.

Javandel, I., and P.A. Witherspoon, A method of analyzing transient fluid flow in multilayered aquifers, Water Resour. Res., 5(4), 856-869, 1969.

McCracken, D.D., and W.S. Dorn, Numerical Methods and Fortran Programming, 457pp., John Wiley, New York, 1964.

Neumann, S.P., and P.A. Witherspoon, Theory of flow in a confined two aquifer system, Water Resour. Res., 5(4), 803-816, 1969.

Neumann, S.P., and P.A. Witherspoon, Flow in multiple aquifer systems, in Sea-Water Intrusion: Aquitards in the Coastal Groundwater Basin of Oxnard Plain, Ventura County, Calif. Dept. of Water Resources Bull. No. 63-4, 21-36, 1971.

Pinder, G.F., and J.D. Bredehoeft, Application of the digital computer for aquifer evaluation, Water Resour. Res., 4(5), 1069-1094, 1968.

Remson, I., G.M. Hornberger, and F.J. Molz, Numerical Methods in Subsurface Hydrology, 389pp., Wiley-Interscience, New York, 1971.

Young, D.M., Jr., The numerical solution of elliptic and parabolic partial differential equations, in Survey of Numerical Analysis, J. Todd, ed., 380-438, McGraw-Hill, New York, 1962.

Prickett, T.A., and C.G. Lonnquist, Selected Digital Computer Techniques for Groundwater Resource Evaluation, Illinois State Water Survey, Bull. No. 55, pp.62, 1971.

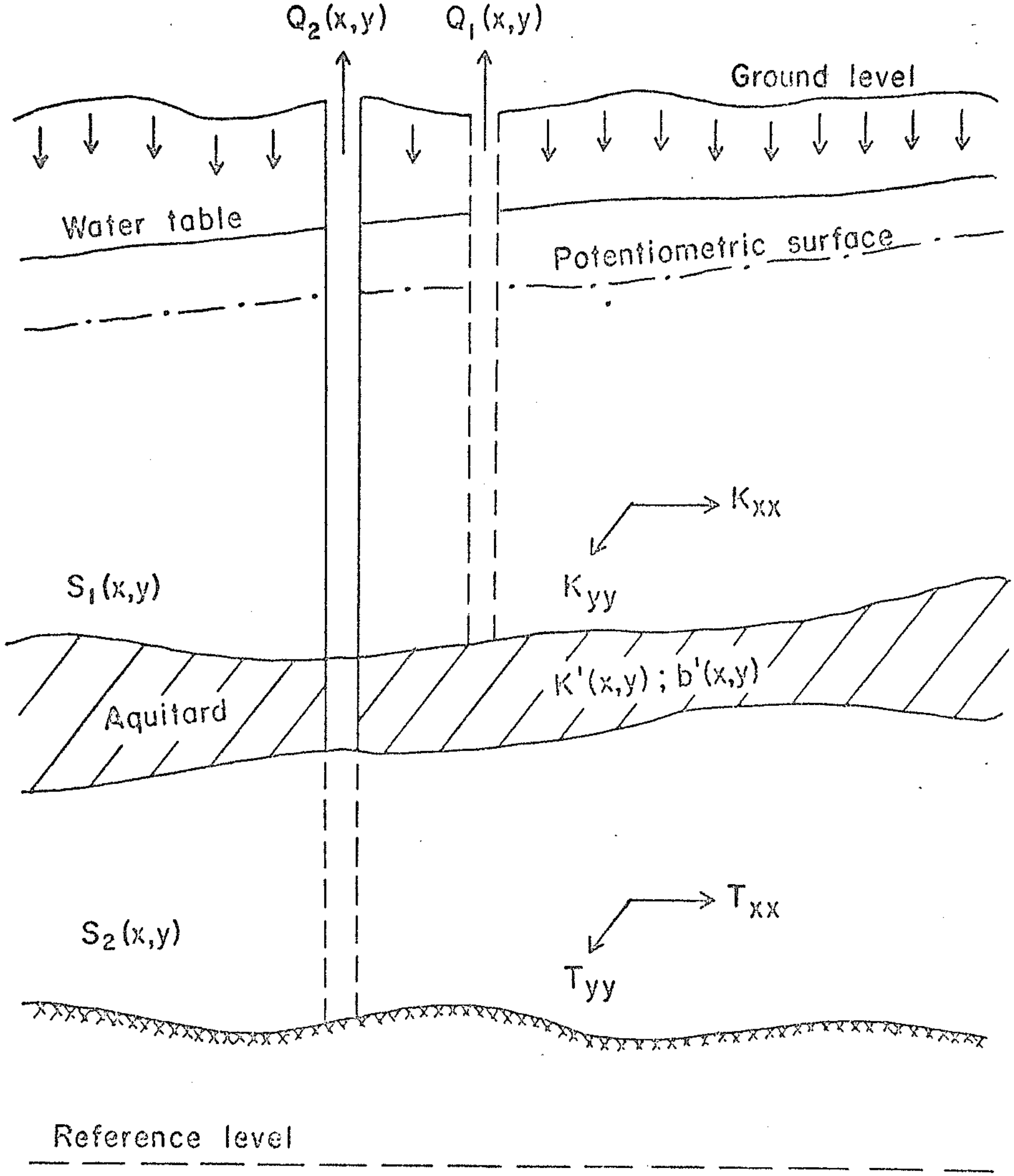

Figure 2-1. Schematic representation of a coupled leaky aquifer system.

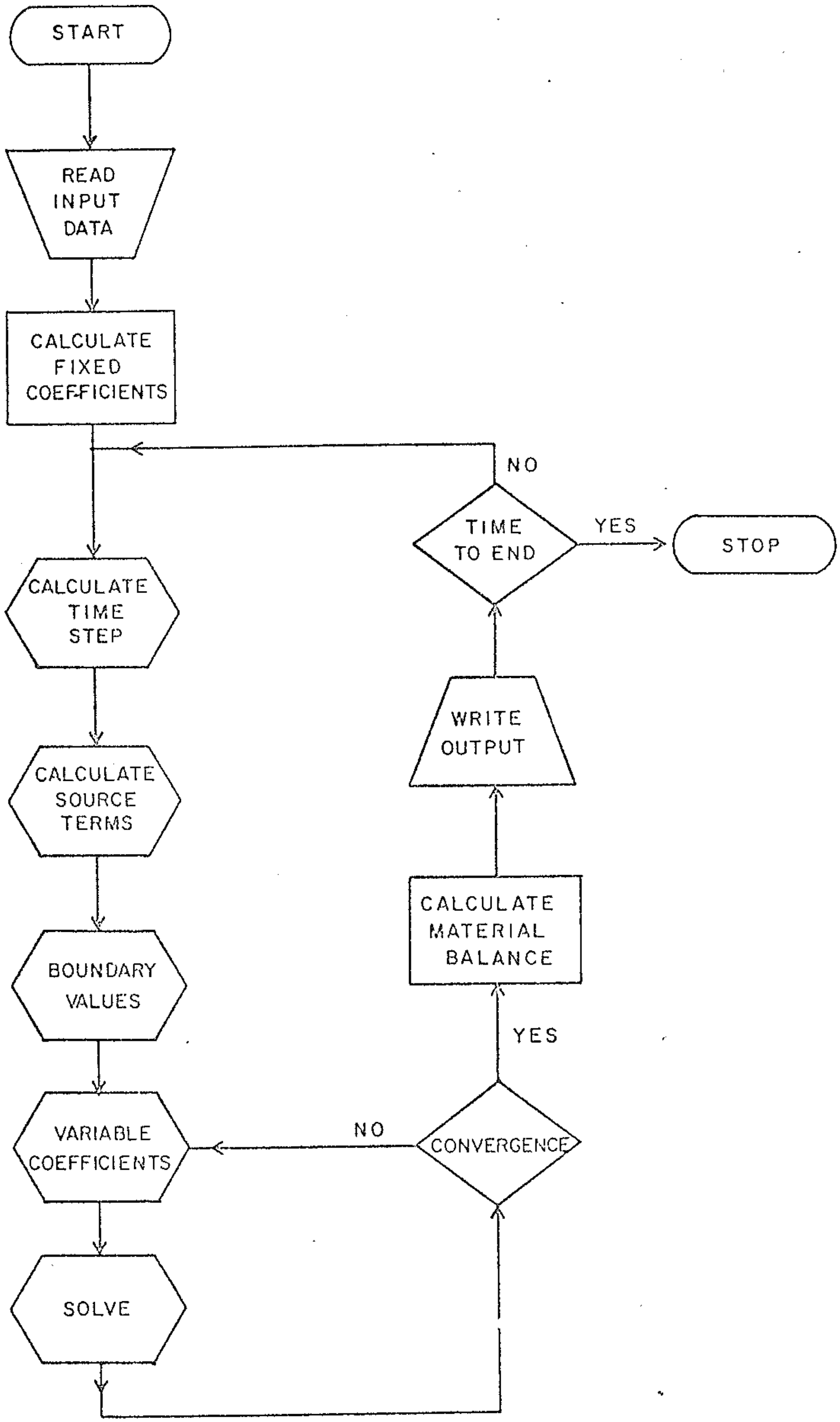

Figure 2-2. Flow chart of computer program.

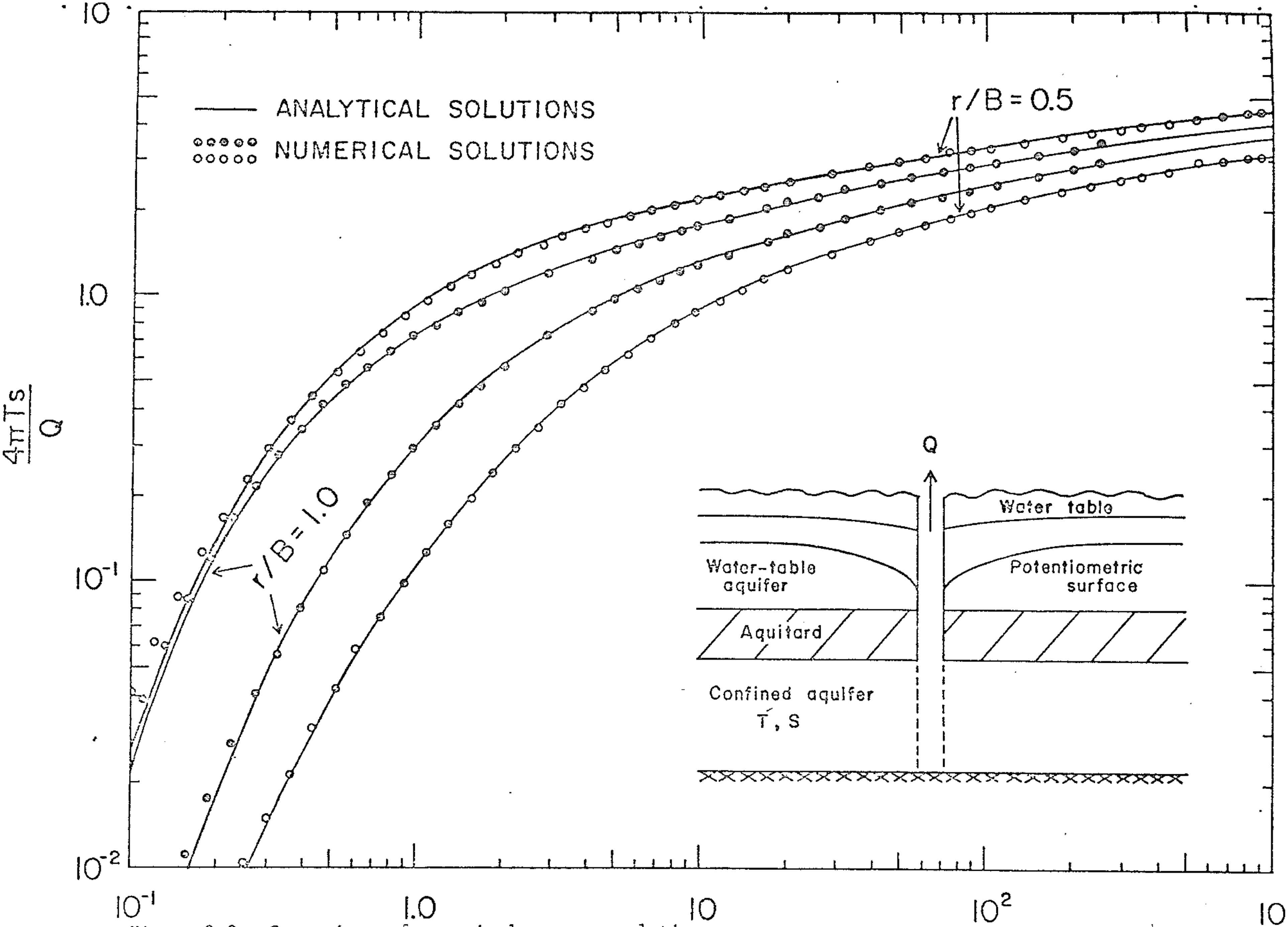

Figure 2-3. Comparison of numerical versus analytical solutions for flow to a well in the confined aquifer of a coupled leaky aquifer system. The lower curves represent induced drawdown in

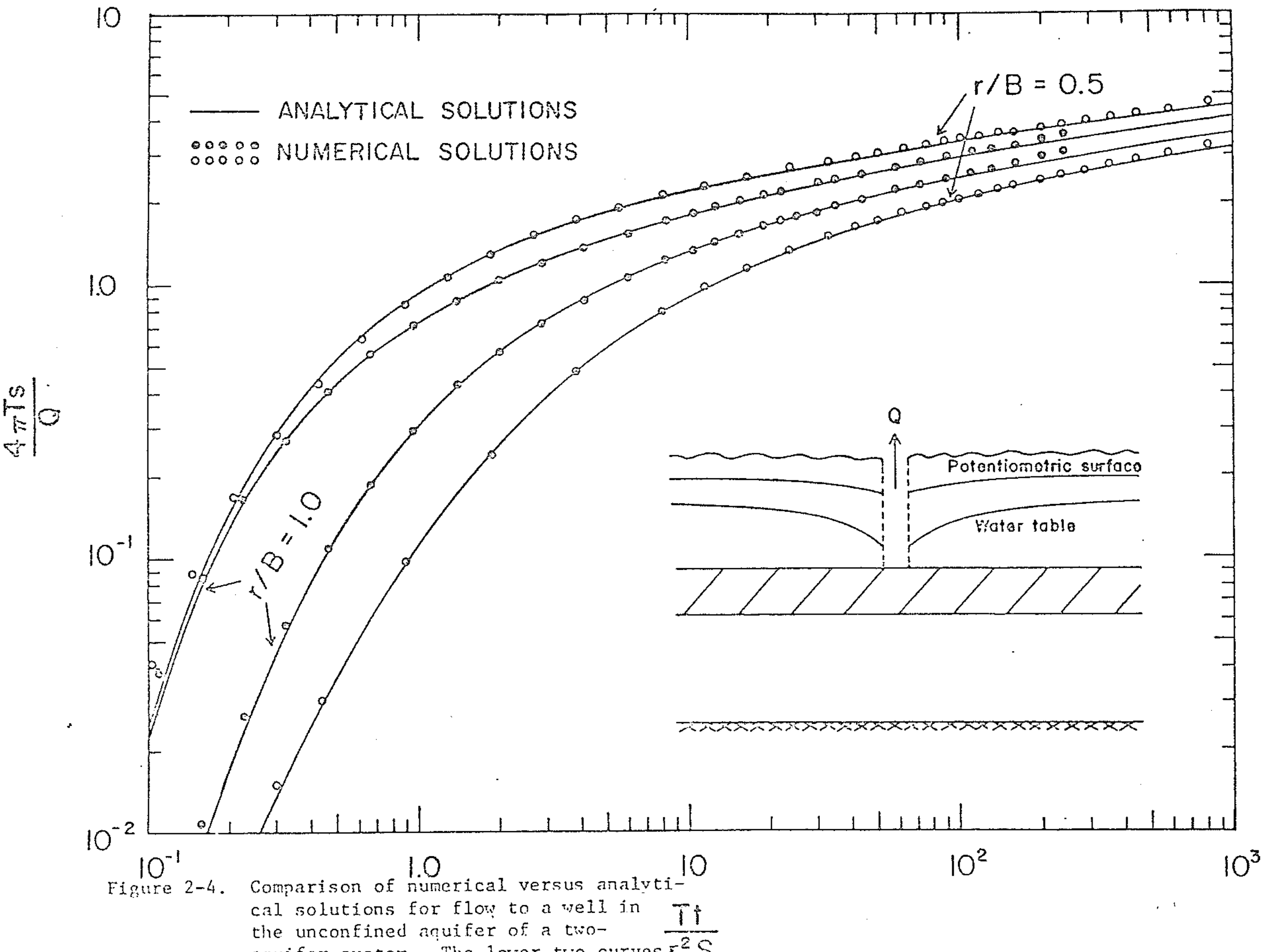

Figure 2-4. Comparison of numerical versus analytical solutions for flow to a well in the unconfined aquifer of a two-aquifer system. The lower two curves represent induced drawdown in the unpumped aquifer.

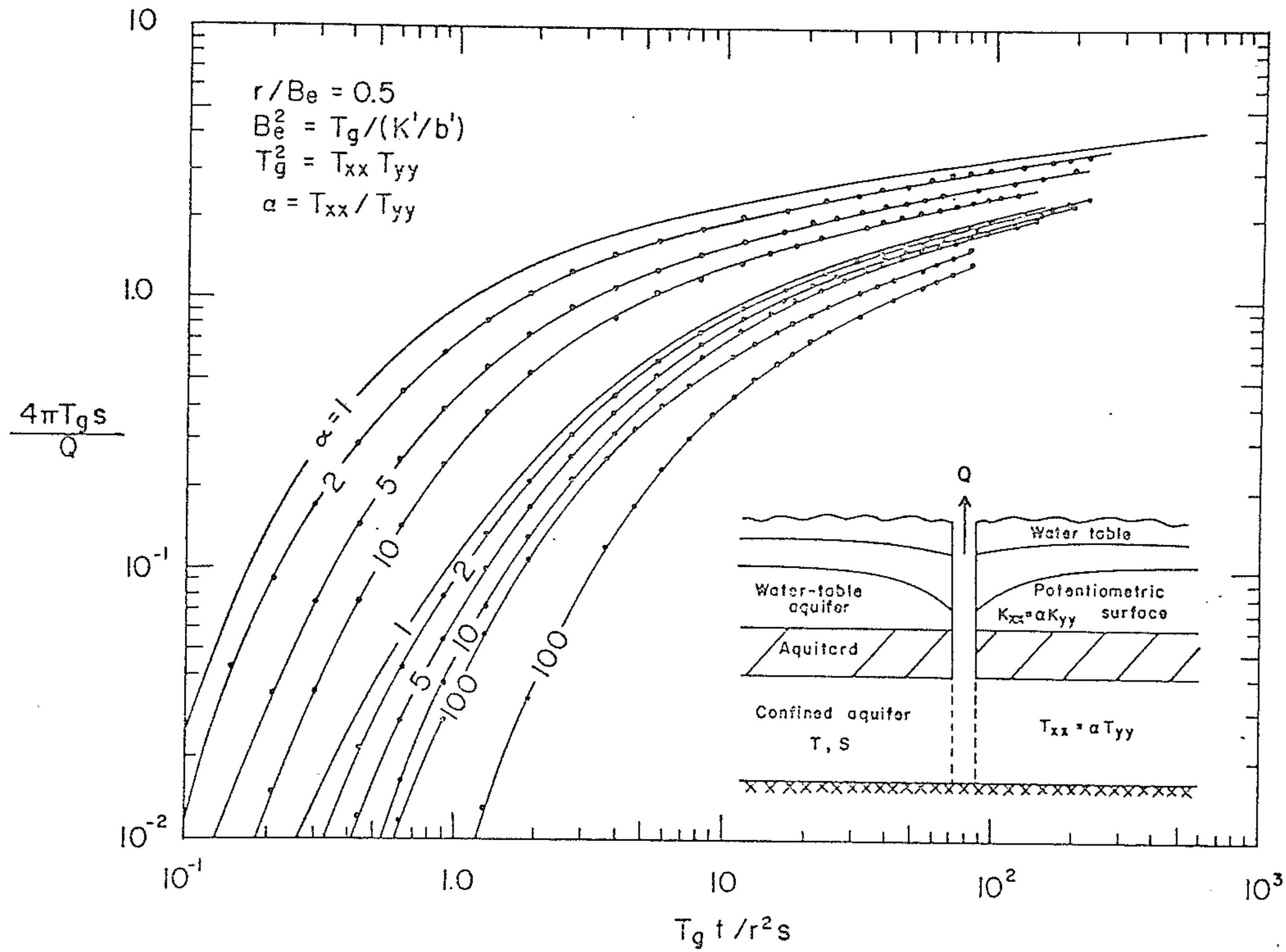

Figure 2-5. Numerical solutions for flow to a well in the confined aquifer of an anisotropic two-aquifer system.

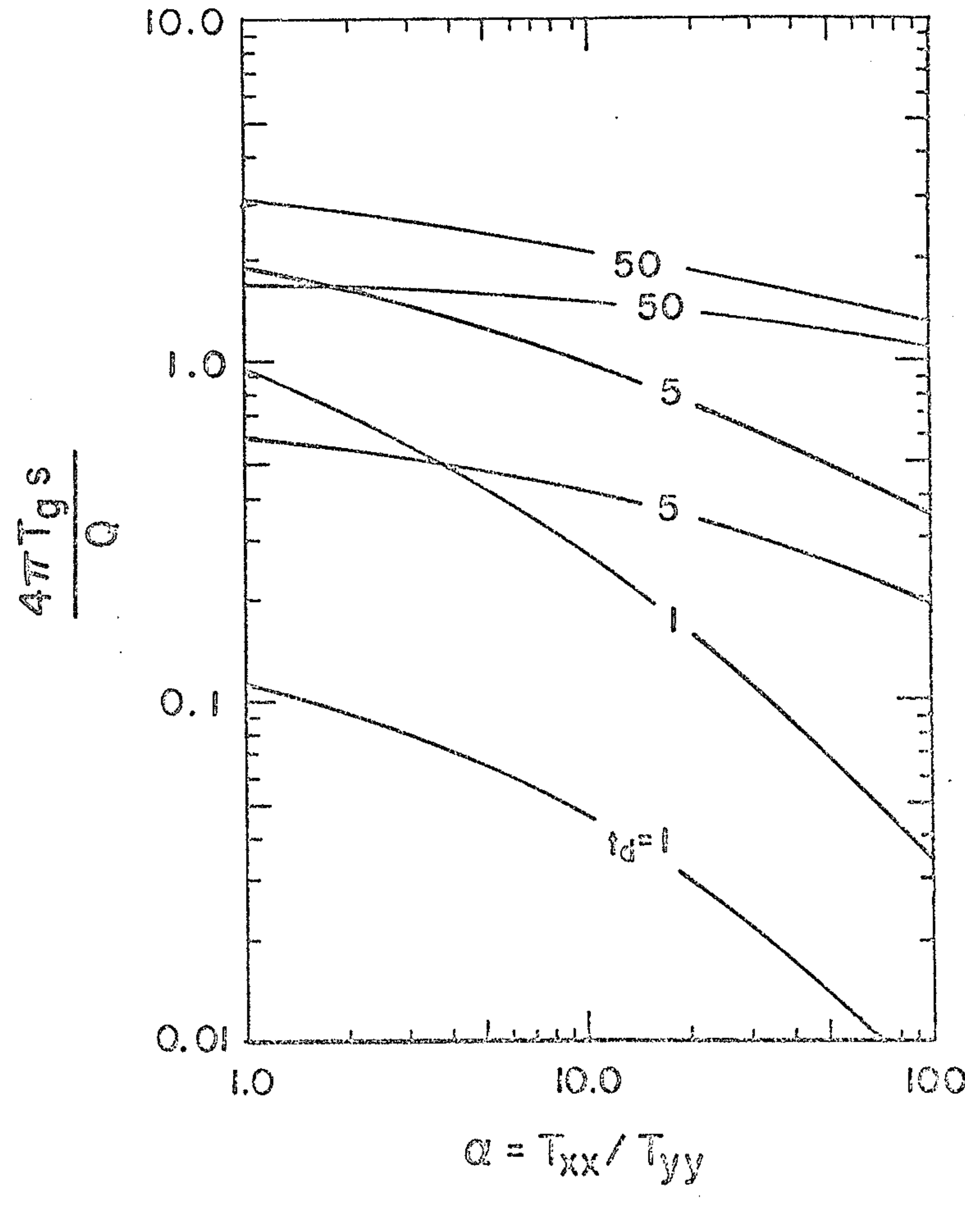

Figure 2-6. Dimensionless drawdown versus anisotropic ratio (=Txx/Tyy) in the two-aquifer system of Figure 5.

Chapter 3

STEADY GROUNDWATER FLOW IN THREE-AQUIFER SYSTEMS

Introduction

Many groundwater basins, containing confined aquifers comprise an intake area and a lowland or valley area. Confined aquifers in such basins receive natural recharge through intake areas, areas where geological formations containing confined aquifers, outcrop or are close to the land surface. Such areas ideally occur in longitudinal strips parallel to, and usually adjoining, perimeters of the groundwater basins. In such basins, the shallow aquifers are in the valley areas overlying the confined aquifers and receive natural recharge directly from precipitation. The shallow and the confined aquifers often are coupled through semipervious layers which permit leakage from one aquifer to the other. Such interacting aquifer systems occur, for example, in the Roswell basin [Saleem and Jacob, 1971] and other areas in New Mexico [Spiegel, 1962].

Analytical solutions of problems in steady and nonsteady flow in two-layered aquifers have been obtained by several investigators [e.g., Hantush and Jacob, 1955; Katz, 1960; Jacquard, 1960; Lefkovitz, 1961; Spiegel, 1962; Polubarinova-Kochina, 1962; Papadopulos, 1966; Hantush, 1967; Saleem and Jacob, 1968; Sternberg, 1968; Neumann and Witherspoon, 1969; Khan et al., 1971]. A few solutions of problems of flow in three-layered aquifer systems have been reported also [Freeze and Witherspoon, 1966; Shahbazi, 1970; Neumann and Witherspoon, 1971; and Toksöz and Kirkham, 1971]. This study concerns the flow of groundwater in a three aquifer system modeled on the Roswell Basin, New Mexico; two aquifers, a deeper confined aquifer and a shallow aquifer, are separated by an aquitard in a layered position and the third aquifer (the intake-area aquifer) is connected to the confined aquifer laterally. The shallow aquifer is drained by a river.

Assumptions and Statement of the Problem

Angles of dip of aquifers in a basin of the type discussed in the last section are often small. Aquifer systems in such basins can be idealized, for mathematical analysis, by a three-aquifer system as shown in Figure 3-1. A shallow aquifer and a deeper confined aquifer are separated by an aquitard which transmits leakage from one aquifer to the other. The direction of leakage at any point in the system is from the aquifer with higher hydraulic head to the aquifer with lower head. The confined aquifer is connected laterally to the intake-area aquifer. The shallow aquifer and the intake-area aquifer can receive steady replenishment due to percolation at different rates which depend on hydrologic characteristics of the system among other factors.

The following assumptions are applicable except where stated otherwise: (1) aquifers individually are homogeneous, isotropic, and of uniform thickness, and the tangents of the angles of tilts are small; (2) the aquifer parameters -- hydraulic conductivities of the aquifers, aquifer thicknesses, and the leakage coefficient of the aquitard -- remain invariant with time and space; (3) the hydraulic conductivities of the shallow and confined aquifers are so large compared with that of the aquitard that the flow in the aquitard is essentially vertical; (4) the spatial variation in head in the intake-area aquifer is small compared with its saturated thickness; (5) the rates of infiltration to the shallow aquifer and the intake-area aquifer are individually uniform and steady; (6) the shallow aquifer loses water as base flow to a perennial river which maintains constant water level.

The problem is to determine the steady-state hydraulic head distribution in the three-aquifer system of Figure 1.

Analysis

Four cases of the three-aquifer system are analyzed in this report. In cases I-A and I-B the vertical variations of the hydraulic head in the shallow unconfined aquifer are so small, relative to its saturated thickness, that the differential equation governing the flow of groundwater in a confined aquifer can be used to describe flow in this aquifer. In cases II-A and II-B, the flow of groundwater in the upper aquifer is represented by an equation which makes use of Dupuit's assumptions [Jacob, 1950] and is modified so that the system of equations is easier to solve. In cases I-A and II-A the confined aquifer at the position of the stream has a known head; in cases I-B and II-B, it has a known flow rate.

Case I-A
(Confined Aquifer Approximation)

Head Known at the Boundary

The steady flow of groundwater in the three-aquifer system of Figure 3-1 is governed approximately by

$$d^2h_1/dx^2-(h_1-h_2)/B_1^2+W_1/T_1 = 0, \qquad (1)$$

$$d^2h_2/dx^2+(h_1-h_2)/B_2^2 = 0, \qquad (2)$$

$$d^2h_3/dx^2+W_3/T_3 = 0, \qquad (3)$$

where the subscripts 1, 2 and 3 stand for the shallow aquifer, the deeper confined aquifer, and the intake-area aquifer, respectively; h, T and W are hydraulic head, transmissivity and rate of deep percolation; and B is the leakage factor

$$B^2 = T/(K'/b'), \qquad (4)$$

in which (K'/b') is the leakage coefficient, K' is the hydraulic conductivity of the aquitard and b' is its thickness.

For case I-A, the boundary conditions are:

$$h_1 = H_1, \text{ when } x = 0; \tag{5}$$

$$h_2 = H_2, \text{ when } x = 0; \tag{6}$$

$$dh_1/dx = 0, \text{ when } x = a; \tag{7}$$

$$h_2 = h_3, \text{ when } x = a; \tag{8}$$

$$T_2 dh_2/dx = K_3 b_2 dh_3/dx, \text{ when } x = a; \tag{9}$$

$$dh_3/dx = 0, \text{ when } x = \ell, \tag{10}$$

where K_3 is the hydraulic conductivity of the intake-area aquifer, b_2 is the thickness of the confined aquifer, a is the distance of the boundary between the confined aquifer and the intake-area aquifer from the stream and ℓ is the width of the basin from the stream to the outer boundary of the intake-area aquifer.

Solution of the Problem

Equations (1) and (2) are multiplied by B_1^2 and B_2^2, respectively, and added together; after integration, the result is:

$$h_1 + \delta h_2 = c_1 + c_2 x - \frac{W_1}{2T_1} x^2, \tag{11}$$

where $\delta = B_2^2/B_1^2$; c_1 and c_2 are constants of integration.

Equations (1) and (2) are multiplied by B_2^2 and subtracted one from the other; after integration, the solution is:

$$h_1 - h_2 = W_1/(T_1 \beta^2) + c_3 \sinh(\beta x) + c_4 \cosh(\beta x), \tag{12}$$

where $\beta^2 = (1+\delta)/B_2^2$; c_3 and c_4 are constants of integration.

Simultaneous solution of equations (11) and (12), and evaluation of the constants with the use of equations (5) through (10), will ultimately yield

$$h_1(x) = F_1/(\beta^2 B_2^2)+[(\alpha_1 a/B_2^2+F_2/B_1^2)/\beta^2]\cdot x-[\alpha_1/(2\beta^2 B_2^2)]\cdot x^2$$
$$+\{[F_3 \tanh(\beta a)-F_2/(\beta\cosh(\beta a))]/(\beta^2 B_1^2)\}\cdot \sinh(\beta x)$$
$$-[F_3/(\beta^2 B_1^2)]\cdot\cosh(\beta x) \qquad (13)$$

$$h_2(x) = (H_1+\delta H_2)/(\beta^2 B_2^2)-\alpha_1/(\beta^4 B_2^2)+[(\alpha_1 a/B_2^2+F_2/B_1^2)/\beta^2]\cdot x$$
$$-[\alpha_1/(2\beta^2 B_2^2)]\cdot x^2- [F_3 \tanh(\beta a)$$
$$-F_2/(\beta\cosh(\beta a))]/(\beta^2 B_2^2)\}\cdot\sinh(\beta x)$$
$$+[F_3/(\beta^2 B_2^2)]\cdot\cosh\beta x \qquad (14)$$

$$h_3(x) = \alpha_1 a^2/(2\beta^2 B_2^2)+[F_2/(\beta^2 B_2^2)][a\delta+\tanh(\beta a)/\beta]$$
$$+F_3/(\beta^2 B_2^2\cosh(\beta a))+(H_1+\delta H_2)/(\beta^2 B_2^2)-\alpha_1/(\beta^4 B_2^2)$$
$$+\alpha_2(\ell-a)^2/2-\alpha_2(\ell-x)^2/2, \qquad (15)$$

where

$$\alpha_1 = W_1/T_1; \text{ and } \alpha_2 = W_2/T_3, \qquad (16)$$

$$\beta = \sqrt{1+\delta}/B_2, \qquad (17)$$

$$F_1 = H_1+\delta H_2+\delta\alpha_1/\beta^2, \qquad (18)$$

$$F_2 = K_3 b_2 \alpha_2(\ell-a)/T_2, \qquad (19)$$

$$F_3 = H_2-H_1+\alpha_1/\beta^2. \qquad (20)$$

The rate of seepage per unit length from the shallow aquifer to the stream is

$$q_1(0) = T_1(dh_1/dx)_{x=0}$$

$$\text{or } q_1(0) = T_1\{\alpha_1 a/(B_2^2\beta^2)+F_3\tanh(\beta a)/(B_1^2\beta)$$

$$+F_2[1-1/\cosh(\beta a)]/(B_1^2\beta^2)\}. \qquad (21)$$

Similarly, the lateral outflow per unit length from the confined aquifer, at x=0, is

$$q_2(0) = T_2\{\alpha_1 a/(B_2^2\beta^2)-F_3\tanh(\beta a)/(B_2^2\beta)$$

$$+F_2[\delta+1/\cosh(\beta a)]/(B_2^2\beta^2)\}. \qquad (22)$$

The inflow per unit length from the intake-area aquifer to the confined aquifer (when x=a) is

$$q_3(a) = K_3 b_2 \alpha_2(\ell-a) = F_2 T_2. \qquad (23)$$

Case I-B
(Confined Aquifer Approximation)

Discharge Known at the Boundary

The flow problem of case I-B is identical to the problem of case I-A, except that now the boundary condition given by equation (6) is replaced by

$$dh_2/dx = f, \text{ when } x = 0, \qquad (24)$$

where f is the rate of discharge from the confined aquifer per unit length per unit transmissivity of the aquifer.

Solution of the Problem

The solution of the problem, represented by equations (1) through (10) with equation (24) replacing equation (6), is the same as for the previous case, except that constants of integration are different.

$$h_1(x) = H_1+F_2F_4\delta/[\beta(1+\delta)]+F_5\alpha_1 a\delta/(1+\delta)-f\delta\coth(\beta a)/\beta$$

$$+B_2^2\alpha_1\delta/(1+\delta)^2+\{[\alpha_1 a+F_2\delta]/(1+\delta)\}\cdot x-\{\alpha_1/[2(1+\delta)]\cdot x^2$$

$$+\{F_2\delta/[\beta(1+\delta)]\}\cdot\sinh(\beta(a-x))-\{\alpha_1 a/[\beta\sinh(\beta a)]$$

$$+[F_2/(\beta\sinh(\beta a))][\delta+\cosh(\beta a)]$$

$$-f(1+\delta)/[\beta\sinh(\beta a)]\}[\delta/(1+\delta)]\cdot\cosh(\beta(a-x)) \qquad (25)$$

$$h_2(x) = H_1+F_2F_4\delta/[\beta(1+\delta)]+F_5\alpha_1 a\delta/(1+\delta)$$

$$-f\delta\coth(\beta a)/\beta-B_2^2\alpha_1/(1+\delta)^2+[(\alpha_1 a+F_2\delta)/(1+\delta)]\cdot x$$

$$-[\alpha_1/(2(1+\delta))]\cdot x^2-[F_2/(\beta(1+\delta))]\cdot\sinh(\beta(a-x))$$

$$+\{\alpha_1 a/(\beta\sinh(\beta a))+F_2[\delta+\cosh(\beta a)]/[\beta\sinh(\beta a)]$$

$$-f(1+\delta)/(\beta\sinh(\beta a))\}\cdot\cosh(\beta(a-x))/(1+\delta) \qquad (26)$$

$$h_3(x) = H_1+[F_2\delta/(\beta(1+\delta))]\cdot[a\beta+F_4+(\delta+\cosh(\beta a))/(\delta\sinh(\beta a))]$$

$$+[\alpha_1 a/(1+\delta)][\delta\coth(\beta a)/\beta-B_2^2/a+a/2+1/(\beta\sinh(\beta a))]$$

$$-(f/\beta)[\delta\coth(\beta a)+1/\sinh(\beta a)$$

$$+\alpha_2(\ell-a)^2/2-(\ell-x)^2/2 \qquad (27)$$

where

$$F_4 = [\delta+\cosh(\beta a)]\cdot\coth(\beta a)-\sinh(\beta a), \qquad (28)$$

$$F_5 = \coth(\beta a)/\beta-B_2^2/[a(1+\delta)]. \qquad (29)$$

The rates of seepage per unit length for this case are

$$q_1(0) = T_1[\alpha_1 a+F_2\delta-f\delta] \qquad (30)$$

and

$$q_3(a) = K_3 b_2\alpha_2(\ell-a) = F_2T_2. \qquad (31)$$

The rate of inflow to the confined aquifer from the intake-area aquifer is the same for this case as for case I-A, which shows that the inflow is independent of the nature of the boundary at x = 0, and of the leakage coefficient. In addition to the rate of percolation, it depends on the hydraulic conductivity and the geometry of the intake-area aquifer.

Case II-A
(Average Saturated Thickness Approximation)

Head Known at the Boundary

The approximate differential equation governing the steady flow of groundwater in the unconfined leaky aquifer of Figure 3-1 can be written as [Hantush, 1964]:

$$d^2h_1^2/dx^2-2K'(h_1-h_2)/(K_1b')+2W_1/K_1=0 \quad (32)$$

Equation (32) is nonlinear in h_1 and is, therefore, rather difficult to solve. However, if h_1 in the second term is replaced by (h_1^2/D), where D is the weighted average depth of the flow profile, the resultant approximate equation is easier to solve. The system of differential equations describing the steady flow in the three-aquifer system of Figure 1 is now:

$$d^2h_1^2/dx^2-(h_1^2/D-h_2)/\beta_1^2+2W_1/K_1 = 0 \quad (33)$$

$$d^2h_2/dx^2+(h_1^2/D-h_2)/B_2^2 = 0, \quad (34)$$

where

$$\beta_1^2 = K_1b'/(2K'),$$

and the aforementioned approximation was applied also to the equation of flow for the confined aquifer, equation (2), to obtain equation (34). Hantush [1962] used a similar approximation and according to him, the approximation is valid provided that extreme values of h_1 in a flow system do not vary greatly from this mean depth ($|D-h_1|<0.25h_1$). The equation of flow for the intake-area aquifer, equation (3), is assumed the same for cases II-A and II-B as for cases I-A and I-B. The boundary conditions for cases II-A and II-B are the same as for cases I-A and I-B, respectively.

Solution of the Problem

The solution for case II-A is obtained by following the identical approach as for case I-A:

$$h_1^2(x) = \{H_1^2+\gamma H_2+2\alpha_1\gamma/(D\mu^2)+(\gamma F_2+2\alpha_1 a)\cdot x-\alpha_1 x^2+[F_6\tanh(\mu a) -\gamma F_2/(\mu\cosh(\mu a))]\cdot\gamma\sinh(\mu x)-\gamma F_6\cosh(\mu x)\}/(\mu^2 B_2^2) \quad (35)$$

$$h_2(x) = \{H_1^2+\gamma H_2-2\alpha_1/\mu^2+(\gamma F_2+2\alpha_1 a)\cdot x-\alpha_1 x^2-[F_6\tanh(\mu a) -F_2/(\mu\cosh(\mu a))]\cdot D\sinh(\mu x)+F_6 D\cosh(\mu x)\}|(D\mu^2 B_2^2) \quad (36)$$

$$h_3(x) = \{H_1^2+\gamma H_2+F_6 D/\cosh(\mu a)+F_2[\gamma a+D\tanh(\mu a)/\mu] +\alpha_1(a^2-2/\mu^2)\}/(D\mu^2 B_2^2)+\alpha_2(\ell-a)^2/2-\alpha_2(\ell-x)^2/2 \quad (37)$$

where

$$\mu = \sqrt{1+\gamma/D}/B_2 \quad (38)$$

and

$$F_6 = H_2-H_1^2/D+2\alpha_1/(D\mu^2). \quad (39)$$

The outflows per unit length for case II-A are

$$q_1(0) = \{2\alpha_1 a+F_6\gamma\mu\tanh(\mu a)+\gamma F_2[1-1/\cosh(\mu a)]\}\cdot K_1/(2\mu^2 B_2^2), \quad (40)$$

$$q_2(0) = \{2\alpha_1 a+F_2[\gamma+D/\cosh(\mu a)]-F_6 D\mu\tanh(\mu a)\}\cdot T_2/(D\mu^2 B_2^2) \quad (41)$$

$$q_3(0) = K_3 b_2\alpha_2(\ell-a) = F_2 T_2. \quad (42)$$

Case II-B
(Average Saturated Thickness Approximation)

Discharge Known at the Boundary

The system of differential equations describing the flow of groundwater for case II-B is the same as for case II-A and boundary conditions are the same as for case I-B.

Solution of the Problem

Solutions for the case II-B are obtained as in case I-B.

$$h_1^2(x) = \{H_1^2\mu^2B_2^2+2\alpha_1\gamma/(D\mu^2)-f\mu B_2^2\gamma\coth(\mu a)$$

$$+[2\alpha_1 a\gamma/(D\mu)]\cdot[\coth(\mu a)-1/(\mu a)]+F_2F_7\gamma/\mu$$

$$+(2\alpha_1 a+F_2\gamma)\cdot x-\alpha_1 x^2+(F_2\gamma/\mu)\cdot\sinh(\mu(a-x))$$

$$+[f\mu B_2^2/(\sinh(\mu a))-2\alpha_1 a/(D\mu\sinh(\mu a))$$

$$-F_2(\gamma/D+\cosh(\mu a))/(\mu\sinh(\mu a))]\cdot\gamma\cosh(\mu(a-x))\}/(\mu^2B_2^2) \qquad (43)$$

$$h_2(x) = \{H_1^2\mu^2B_2^2-f\mu B_2^2\gamma\coth(\mu a)+[2\alpha_1 a\gamma/(\mu D)]\cdot[\coth(\mu a)-1/(\mu a)]$$

$$-2\alpha_1/\mu^2+F_2F_7\gamma/\mu+(2\alpha_1 a+F_2\gamma)\cdot x-\alpha_1 x^2-(F_2D/\mu)\cdot\sinh(\mu(a-x))$$

$$-[f\mu B_2^2-2\alpha_1 a/(D\mu)-F_2(\gamma/D+\cosh(\mu a))/\mu]\cdot(D/\sinh(\mu a))\cdot$$

$$\cosh(\mu(a-x))\}/(\mu^2B_2^2D) \qquad (44)$$

$$h_3(x) = \{H_1^2\mu^2B_2^2-f\mu B_2^2(\gamma\cosh(\mu a)+D/\sinh(\mu a)$$

$$+(F_2/\mu)\cdot[\gamma a\mu+\gamma F_7+D(\gamma/D+\cosh(\mu a))/\sinh(\mu a)]$$

$$+(2\alpha_1 a)\cdot[1/(\mu\sinh(\mu a)-1/(a\mu^2)+a/2+(\gamma/D\mu)\cdot(\coth(\mu a)$$

$$-1/(a\mu))]\}/(\mu^2B_2^2D)+\alpha_2(\ell-a)^2/2-\alpha_2(\ell-x)^2/2 \qquad (45)$$

where

$$F_7 = [\gamma/D+\cosh(\mu a)]\cdot\coth(\mu a)-\sinh(\mu a) \quad (46)$$

Outflows per unit length from the unconfined aquifer and the intake-area aquifer are

$$q_1(0) = K_1[\alpha a+F_2\gamma/2] \quad (47)$$

and $$q_3(a) = K_3 b_2 \alpha_2 (\ell-a) = F_2 T_2 \quad (48)$$

Results

The solutions derived for cases I-A, I-B, and II-B were programmed for evaluation on the IBM 370, Model 155, computer. The computer programs are given in the Appendix. A sensitivity analysis was performed to assess effects of different hydraulic characteristics on the flow in the three-aquifer system. Table 3-1 shows the different values of aquifer parameters used in the computer runs. Dimensionless hydraulic heads in the shallow, in the confined, and in the intake-area aquifers were plotted against dimensionless distance from the river and are shown in Figures 3-2 through 3-9 along with values of the leakage and of outflows from the shallow and confined aquifers.

Changes in the flow in the three-aquifer system were studied due to changes in: (1) boundary conditions: (2) transmissivities of the shallow and the confined aquifer; (3) recharge to the shallow and to the intake-area aquifer; (4) The leakage coefficient of the aquitard; and (5) the depth of the flow profile for case II-B. The following results were derived:

1. An increase in head in the confined aquifer at the location of the river ($\underline{H}_2$) increases head values in all three aquifers, increases base flow from the shallow aquifer, decreases outflow from the confined aquifer, and increases leakage which is positive when upward from the confined to the shallow aquifer (Figures 3-2 and 3-3). The effects of a decrease in $\underline{H}_2$ are opposite to that due to the increase.

2. An increase in the discharge from the confined aquifer (Figure 3-5, case I-B) decreases the head in all three aquifers, decreases the outflow from the shallow aquifer, and decreases the leakage from the confined aquifer to the shallow aquifer.

3. An increase in the transmissivity of the shallow aquifer decreases

the head values in all three aquifers, increases the discharge from the shallow aquifer at the river location, and decreases the discharge from the confined aquifer at the same location (see Figure 3-5). An increase in the transmissivity of the confined aquifer decreases the head in all three aquifers, increases the discharge from the confined aquifer, and decreases the discharge from the shallow aquifer.

4. The leakage from the confined aquifer to the shallow aquifer is equal to the inflow from the intake-area aquifer to the confined aquifer minus the outflow from the confined aquifer. Any change in the leakage coefficient changes the leakage which in turn affects the heads and outflows from the system (Figures 3-2, 3-3, 3-7, and 3-8).

5. In general, an increase in recharge to the shallow aquifer and to the intake-area aquifer increases heads and increases outflows from the shallow and the confined aquifers at the location of the stream (Figures 3-2 and 3-7). An increase in recharge to the shallow aquifer decreases leakage and an increase in recharge to the intake-area aquifer increases leakage as expected.

6. Figure 3-9 shows a comparison between cases I-B and II-B. The disagreement is very significant. The hydraulic head in the confined aquifer is lower for case II-B than the corresponding head for case I-B, when the average depth of the flow profile, $\underline{D}$, in the shallow aquifer is higher than the head in the shallow aquifer as given by case I-B. When $\underline{D}$ is lower, the heads given by case II-B are higher than the corresponding head values for case I-B. The head in the intake-area aquifer also follows the above rules. The head in the shallow aquifer is lower for all the runs for case II-B than for case I-B except near the location of the river where they are kept fixed at an equal value.

Table 3-1

Characteristics of the Three-Aquifer System used in derivation of the results shown in Figures 3-2 through 3-9.

Figure number	Run number	Shallow aquifer			Confined aquifer		Aquitard	Intake-area aquifer	Case
		Recharge rate, W_1, ft/day	Transmissivity, T_1, ft^2/day	Head at x = 0, H_1, ft	Transmissivity, T_2, ft^2/day	Head at x = 0, H_2, ft	Leakance K'/b' per day	Recharge rate, W_2, ft/day	
3-2	1	0.00011	14,720	200	13,000	200	1.0×10^{-4}	0.00013	I-A
	2	"	"	"	"	"	"	0.00065	
	3	0.00055	"	"	"	"	"	0.00013	
	4	"	"	"	"	"	"	0.00065	
3-3	1	0.00011	14,720	200	13,000	200	1.0×10^{-4}	0.00013	I-A
	2	"	"	"	"	"	"	0.00065	
	3	0.00055	"	"	"	"	"	0.00013	
	4	"	"	"	"	"	"	0.00065	
3-4	1	0.00011	14,720	200	13,000	200	1.0×10^{-3}	0.00013	I-A
	2	"	"	"	"	"	"	0.00065	
	3	0.00055	"	"	"	"	"	0.00013	
	4	"	"	"	"	"	"	0.00065	
3-5	1	0.00011	14,720	200	13,000	200	1.0×10^{-4}	0.00011	I-A
	2	"	8,920	"	"	"		"	
	3	"	14,720	"	130,000	"		"	
	4	"	"	"	13,000	180		"	

Continued on next page

Table 3-1 (cont.)

		Shallow aquifer			Confined aquifer		Aquitard	Intake-area aquifer	
Figure number	Run number	Recharge rate, W_1, ft/day	Transmissivity, T_1, ft^2/day	Head at x = 0, H_1, ft	Transmissivity, T_2, ft^2/day	Head at x = 0, H_2, ft	Leakance K'/b' per day	Recharge rate, W_2, ft/day	Case
3-6	1	0.00011	14,720	200	13,000	0.0001*	$1.0x10^{-4}$	0.00011	I-B*
	2	"	"	"	"	0.001	"	"	
	3	"	"	"	130,000	0.0	"	"	
	4	"	8,920	"	13,000	0.0	"	"	
3-7	1	0.00011	8,920	200	13,000	0.0001*	$1.0x10^{-4}$	0.00013	I-B*
	2	"	"	"	"	"	"	0.00065	
	3	0.00055	"	"	"	"	"	0.00013	
	4	"	"	"	"	"	"	0.00065	
3-8	1	0.00011	8,920	200	13,000	0.0001*	$1.0x10^{-3}$	0.00013	I-B*
	2	"	"	"	"	"	"	0.00065	
	3	0.00055	"	"	"	"	"	0.00013	
	4	"	"	"	"	"	"	0.00065	
3-9	1	0.00011	40ft/day	210	13,000	0.0	$1.0x10^{-4}$	0.00011	II-B**
	2	"	"	223	"	"		"	
	3	"	"	230	"	"		"	

* In all I-B cases, H_2 is replaced by $(dh_2/dx)_{x = 0}$.

** In case II-B, transmissivity (T_1) is replaced by the hydraulic conductivity (K_1) and the head H_1 is replaced by the average depth of the flow profile, D, in the shallow aquifer.

Conclusions

The roles of various characteristics of the three-aquifer system in controlling the groundwater flow were determined by evaluation of the solutions for the three cases of the three-aquifer system using a digital computer. The leakage coefficient of the aquitard plays a key role in governing the flow in the system. Any variation in the characteristics of any of the aquifers or in the recharge rates to the intake-area aquifer and to the shallow aquifer affects the flow in all three aquifers of the system.

The use of the weighted average depth of the flow profile for the shallow aquifer in obtaining the differential equations for flow (Cases II-A and II-B) facilitates the derivation of analytical solutions. However, the approximation affects the flow distribution in all three aquifers and, therefore, should be used with caution.

The analytical solutions to problems of steady flow in the three-aquifer system derived here can be easily evaluated. The solutions can be employed for the identification of aquifer parameters of three-aquifer systems where water-level data are available at the time systems were being developed. The steady-state solutions, such as derived here, provide not only a tool for the understanding of the flow systems but are also useful because they are needed as initial conditions for problems of unsteady-state groundwater flow.

References

Freeze, R. Allan, and P.A. Witherspoon, Theoretical analysis of regional groundwater flow: 1. Analytical and numerical solutions to the mathematical model, Water Resour. Res., 2(4), 641-656, 1966.

Hantush, M.S., Drainage wells in leaky water-table aquifers, Proc. Amer. Soc. Civil Eng., 88(HY2), 123-137, 1962.

Hantush, M.S., Hydraulics of Wells, in Advances in Hydroscience, V.T. Chow, ed., v. 1. 281-442, Acad. Press, New York, 1964.

Hantush, M.S., Flow to wells in aquifers separated by a semipervious layer, J. Geophys. Res., 7216, 1709-1720, 1967.

Hantush, M.S., and C.E. Jacob, Steady three-dimensional flow to a well in a two-layered aquifer, Trans. Am. Geophys. Union, 36(2), 286-292, 1955.

Jacob, C.E., Flow of groundwater in: Engineering Hydraulics (H. Rouse, ed.), Wiley, N.Y., 1950.

Jacquard, P., Étude mathematique du drainage d'un réservoir hétérogéne, Rev. Inst. Francais du Pétrole, XV, 10, 1384, 1960.

Katz, M.L., Fluid flow and heat transfer in stratified systems, Ph.D. thesis, University of Michigan, Ann Arbor, Michigan, 1960.

Khan, M.Y., Don Kirkham, and S. Tolsoz, Steady state flow around a well in a two-layered aquifer, Water Resour. Res., 7(1), 155-165, 1971.

Lefkovitz, H.C., P. Hazelbrock, E.E. Allen, and C.S. Mathews, A study of the behavior of bounded reservoirs composed of stratified layers, Jour. Soc. Petrol. Engrs., 1, 43-58, 1961.

Neumann, S.P., and P.A. Witherspoon, Theory of flow in a confined two aquifer system, 5(4), 803-816, 1969.

Neumann, S.P., and P.A. Witherspoon, Flow in multiple aquifer systems, in Sea-Water Intrusion: Aquitards in the Coastal Groundwater Basin of Oxnard Plain, Ventura County, Calif. Dept. of Water Resources Bull. No. 63-4, 21-36, 1971.

Papadopulos, S.S., Nonsteady flow to multiaquifer wells, J. Geophys. Res., 71(20), 4791-4797, 1966.

Polubarinova-Kochina, P. Ya., Theory of Groundwater Movement, Translated from Russian by R.J.M. De Wiest, Princeton University Press, Princeton, New Jersey, 1962.

Saleem, Z.A., and C.E. Jacob, Dynamic Programming Model and Quantitative Analysis, Roswell Basin, 180 pp., N. Mex. Water Resour. Inst. Rep. 10, January 1971.

Shahbazi, M., Analytical solutions of the Laplace equation in Regional groundwater flow field, Jour. of Hydraul. Res., 8(4), 457-476, 1970.

Spiegel, Z., Hydraulics of Certain Stream-Connected Aquifer Systems, Special Report, New Mex. State Engineer, Santa Fe, N. Mex., 1962.

Sternberg, Yaron M., Nonsteady two-layer radial flow to wells, Intern. Assoc. Sci. Hdrol. Pub. No. 77, 329-342, 1968.

Toksoz, S., and Don Kirkham, Steady drainage of soils: I, theory, Proc. Amer. Soc. Civil Eng., 97(IR1), 1-18, 1971.

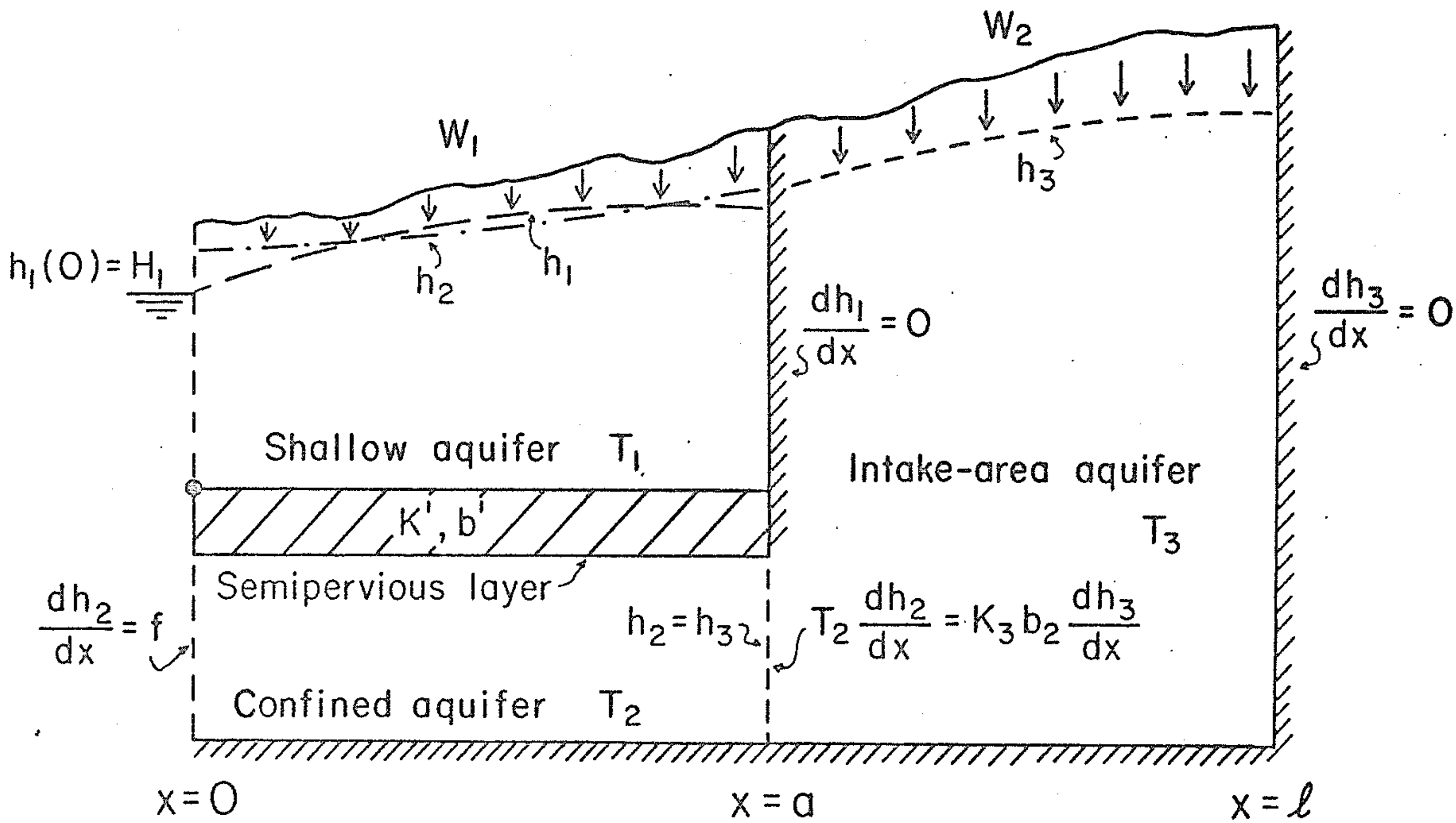

Figure 3-1. Schematic representation of a three-aquifer system.

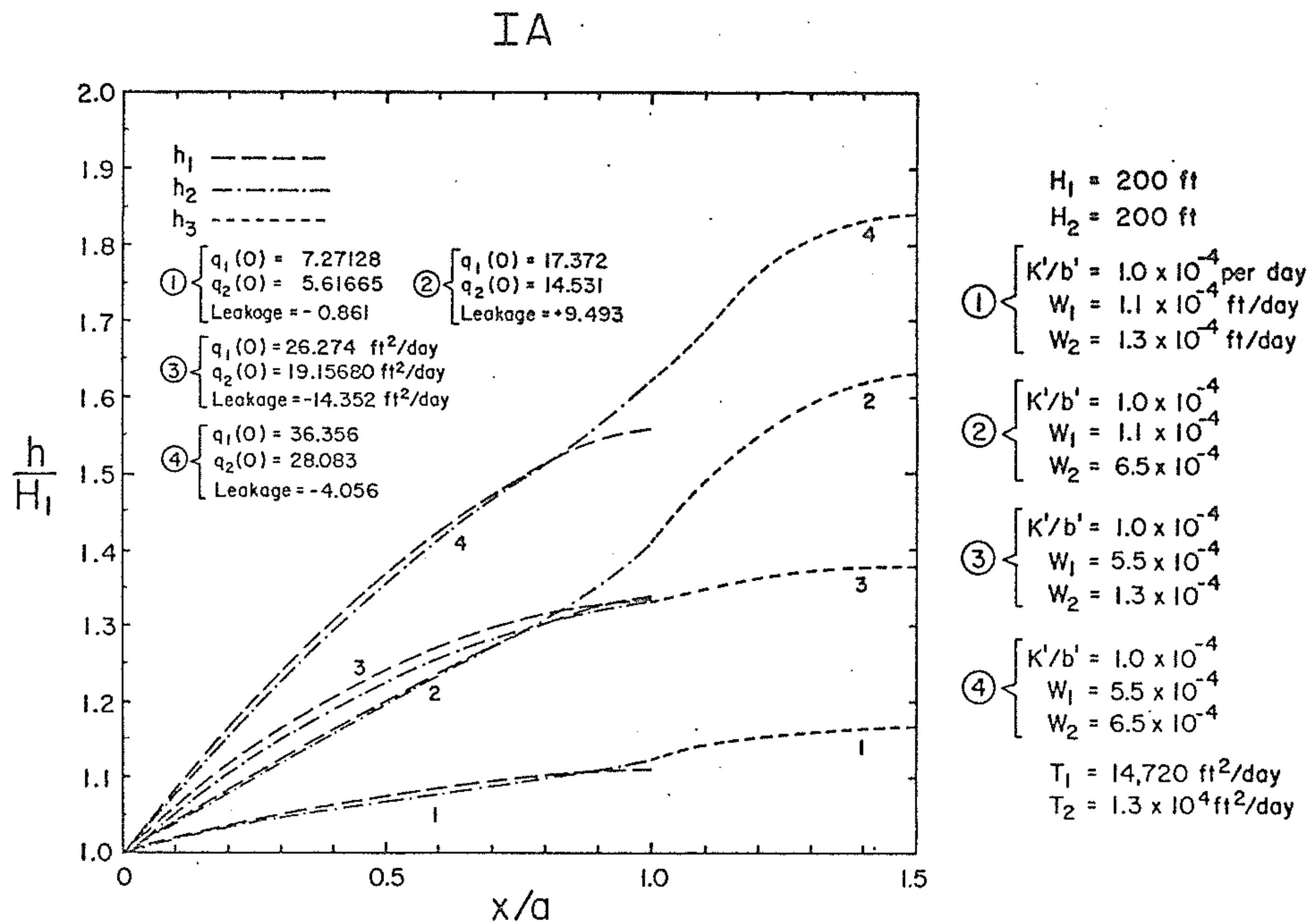

Figure 3-2. Effect of changes in rates of recharge to the intake-area aquifer and the shallow aquifer on groundwater flow in the three-aquifer system ($H_1 = H_2$, K'/b'=1xE-04 per day).

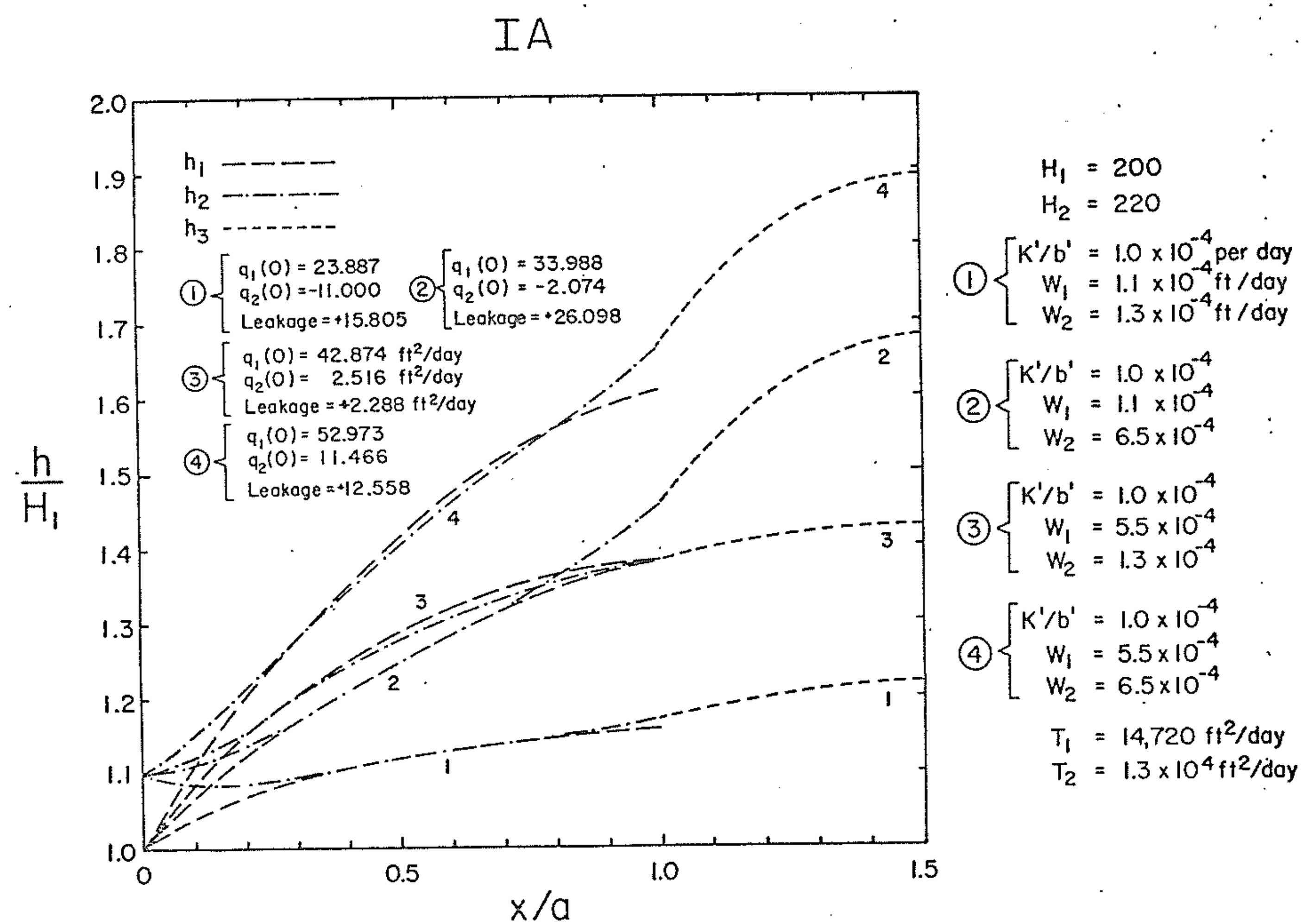

Figure 3-3. Effect of changes in rates of recharge to the intake-area aquifer and the shallow aquifer on groundwater flow in the three-aquifer system ($H_2 > H_1$, K'/b'=1xE-04 per day).

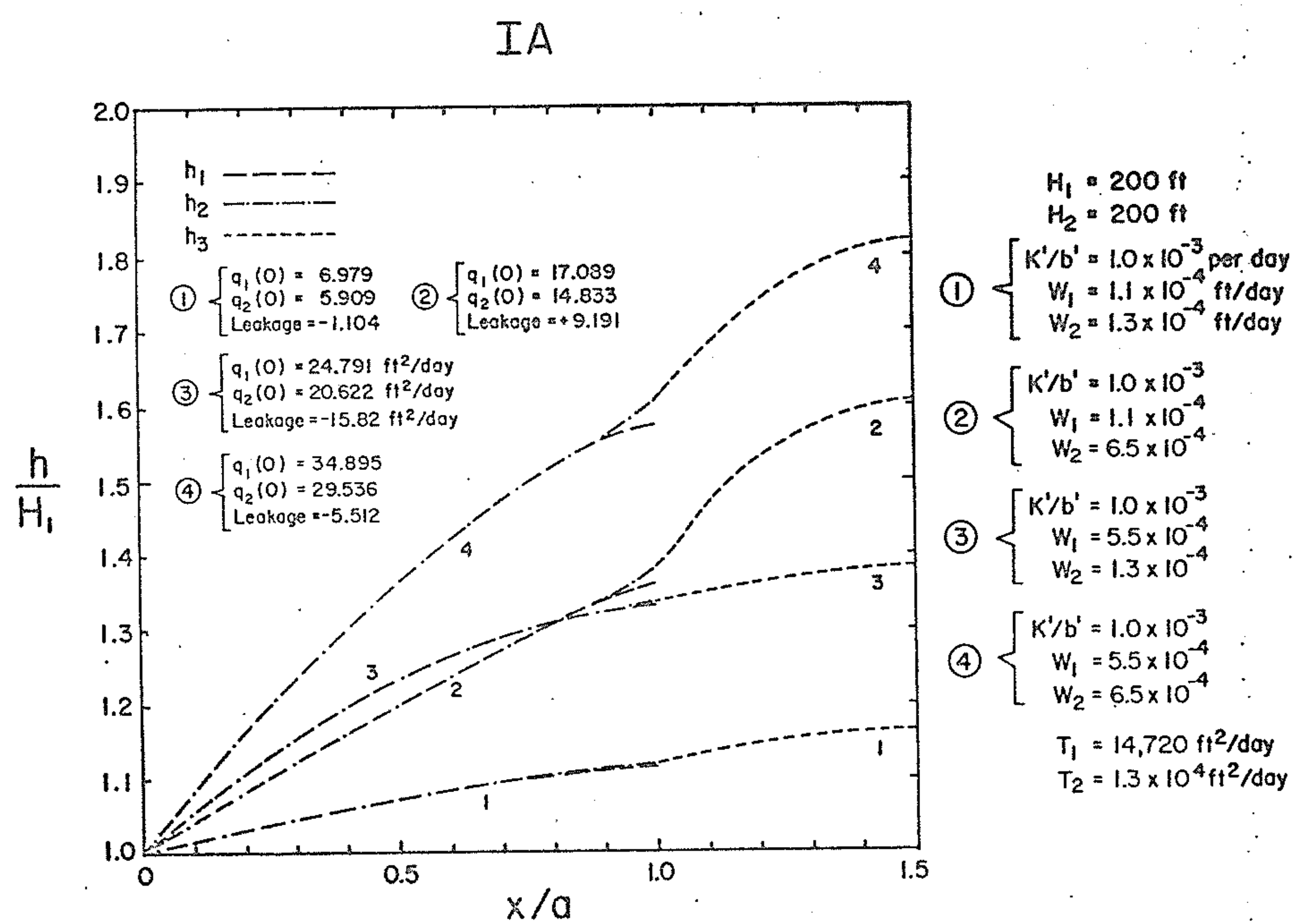

Figure 3-4. Effect of changes in the rates of recharge to the intake-area aquifer and the shallow aquifer on groundwater flow in the three-aquifer system ($H_2=H_1$, K'/b'=1xE-03 per day).

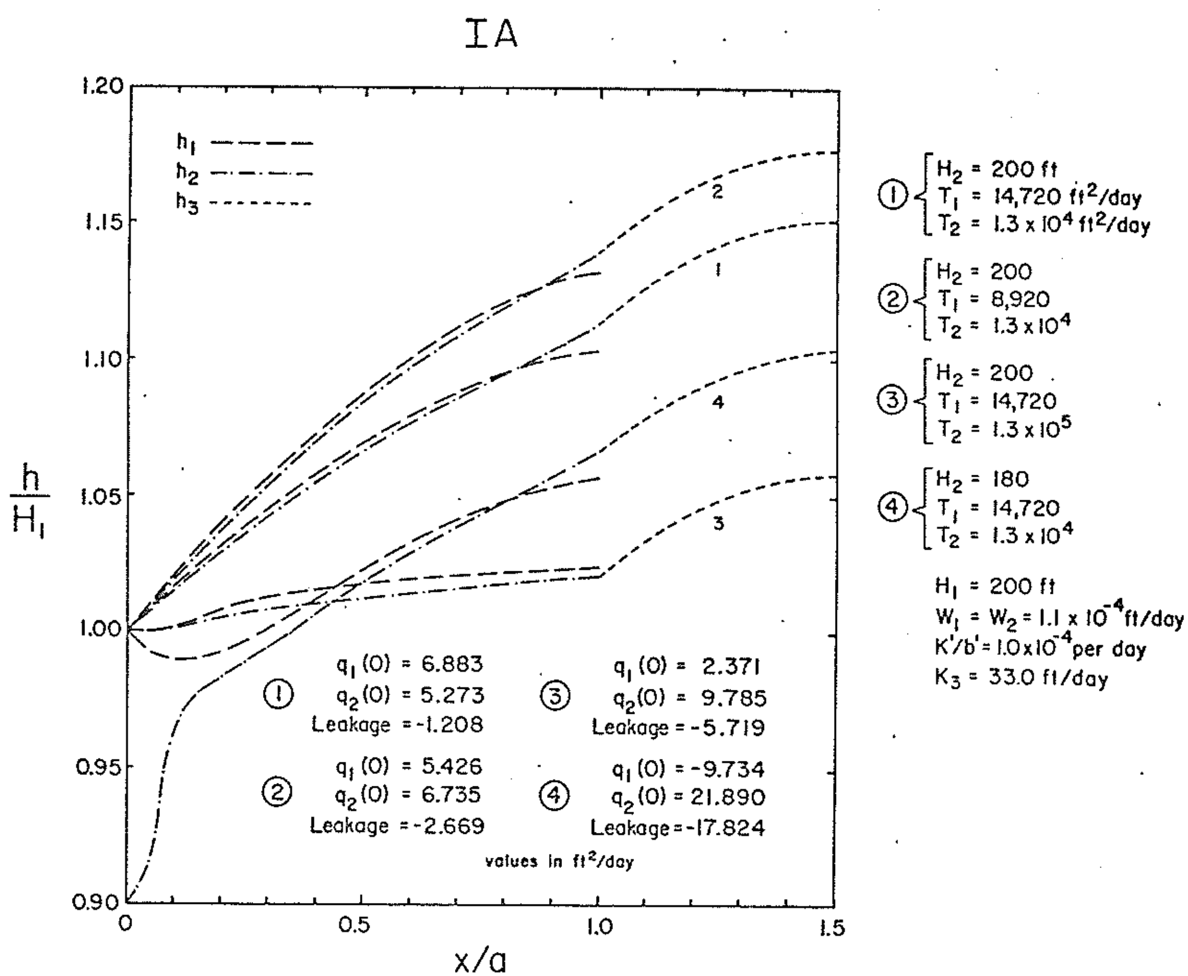

Figure 3-5. Effect of changes in the hydraulic head in the confined aquifer at the location of the stream and changes in transmissivities of the shallow and of the confined aquifer on the flow in the three-aquifer system ($W_1 = W_2$).

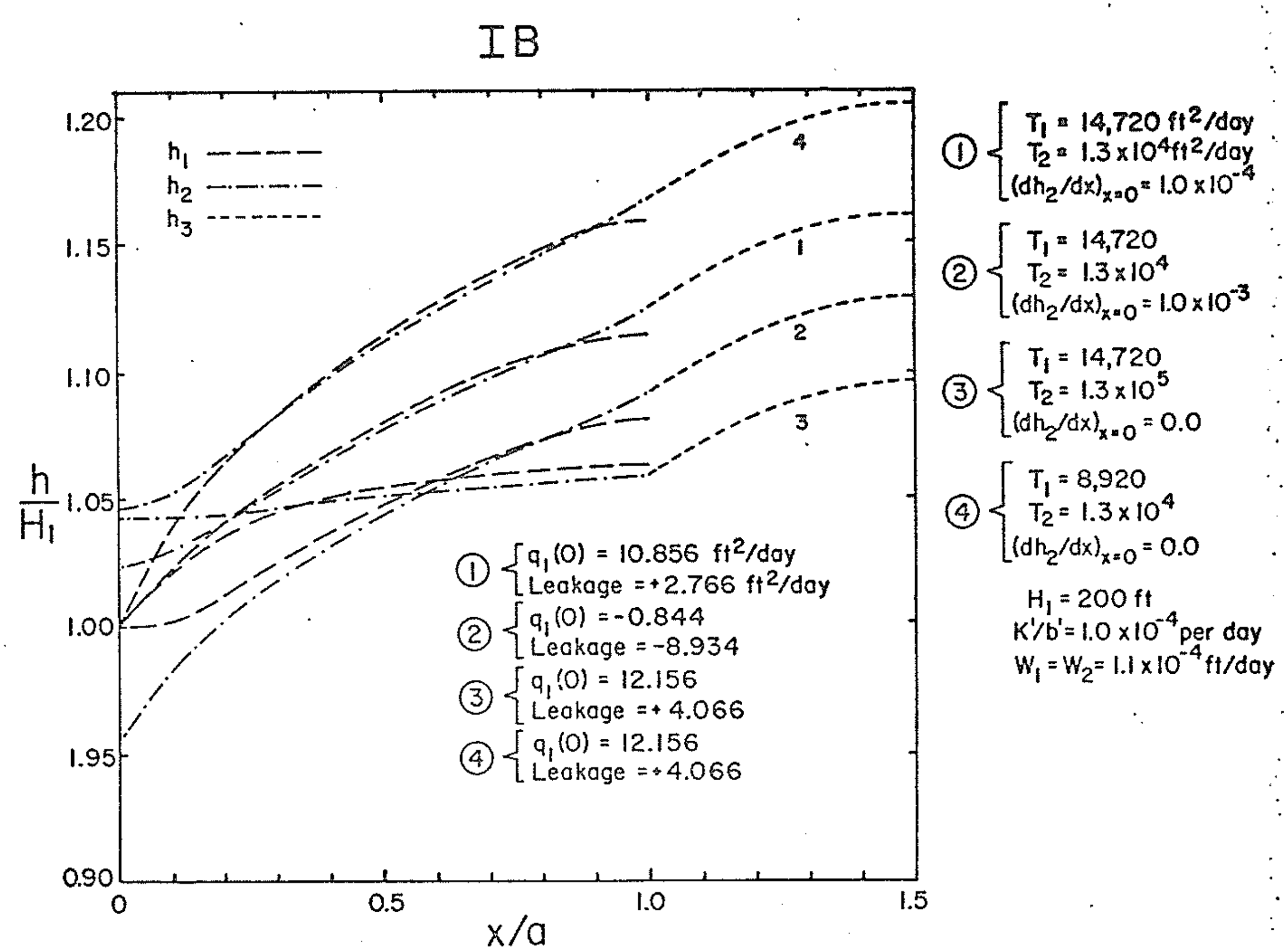

Figure 3-6. Effect of changes in the rates of outflow from the confined aquifer at x=0 and of changes in transmissivities of the shallow and of the confined aquifers on the flow in the three-aquifer system ($W_1 = W_2$).

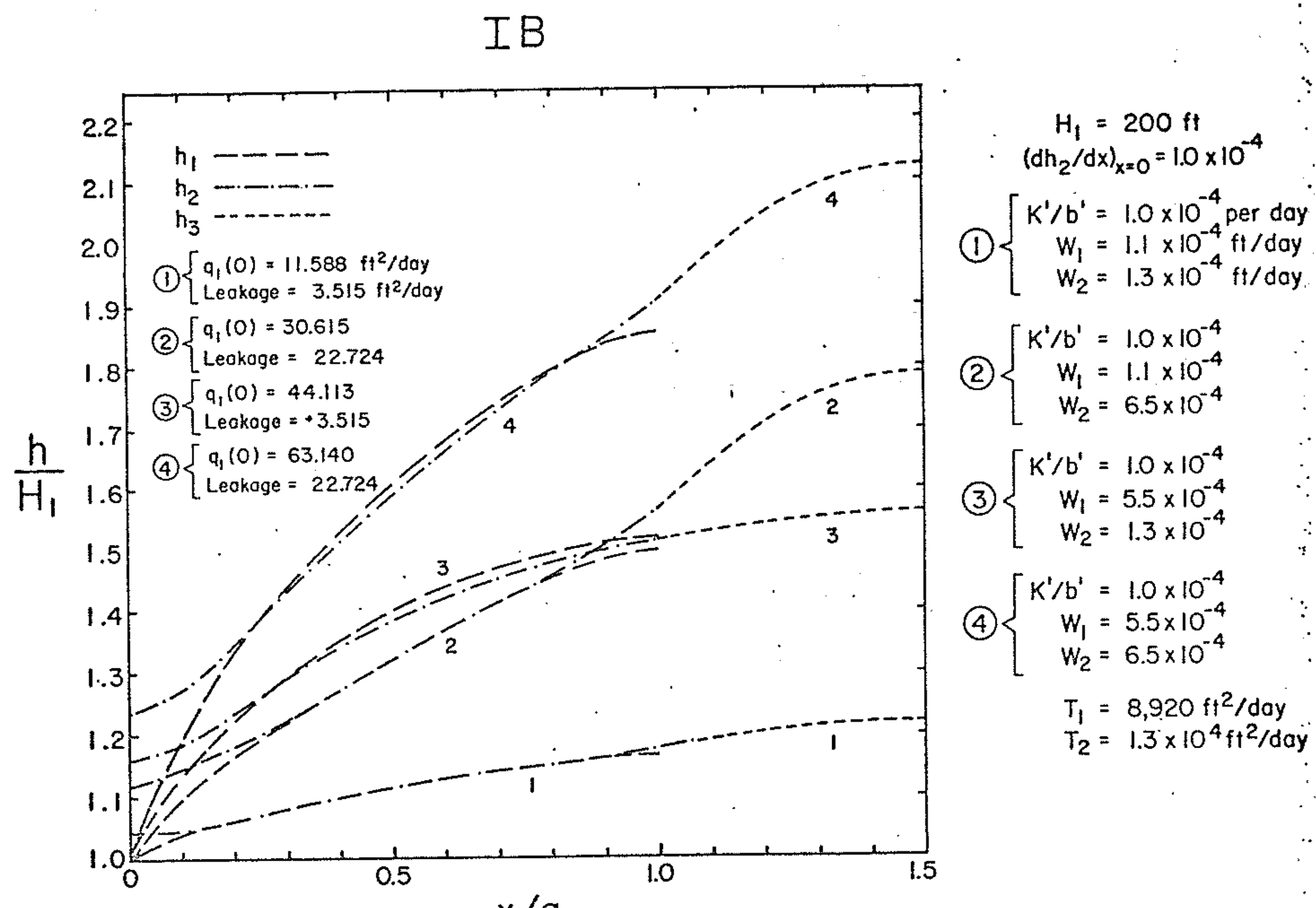

Figure 3-7. Effect of changes in rates of recharge to the shallow and the intake-area aquifers on flow in the three-aquifer system (case I-B).

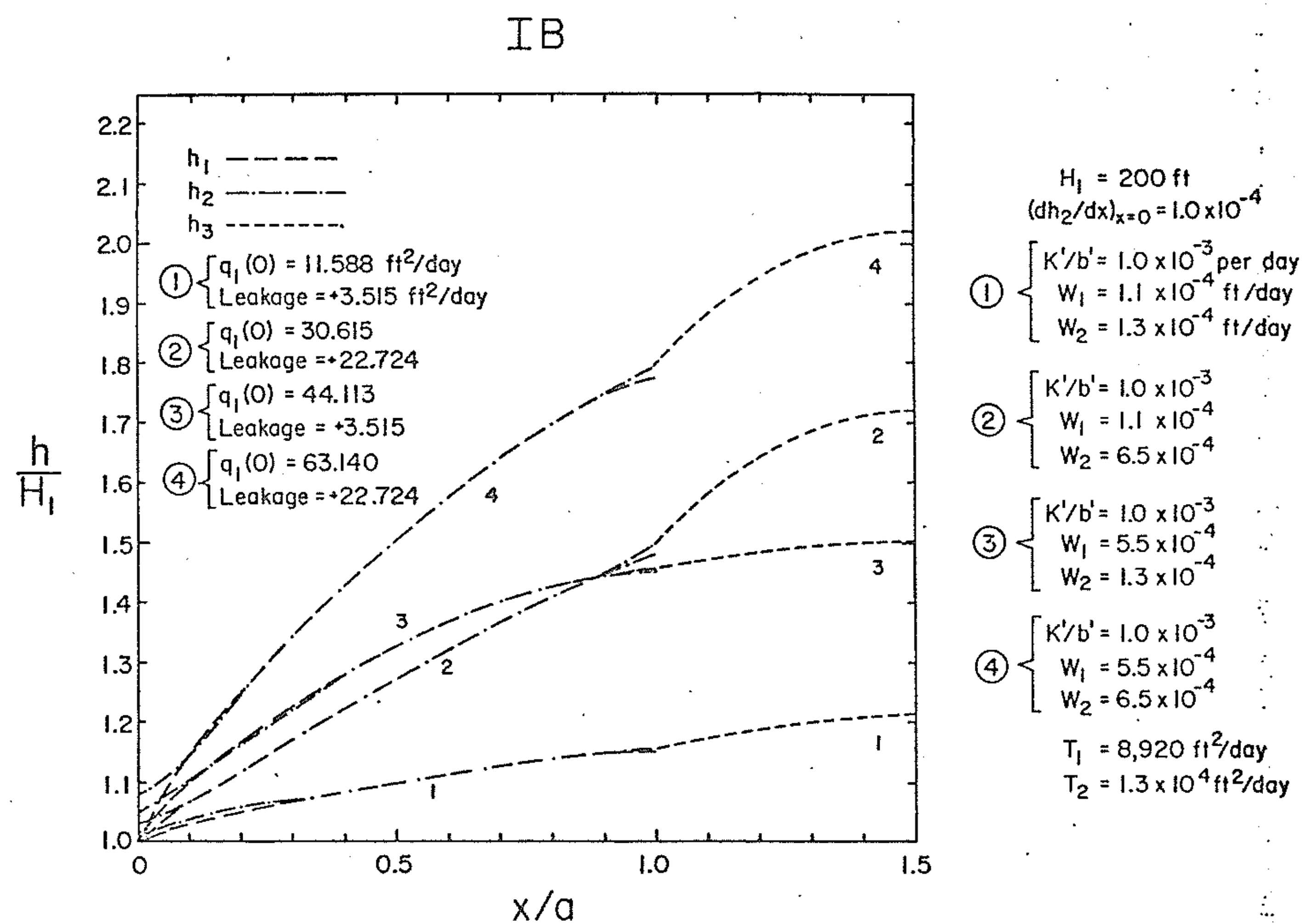

Figure 3-8. Effect of changes in the leakage coefficient of the aquitard and in rates of recharge to the shallow and the intake-area aquifers on flow in the three-aquifer system (case I-B).

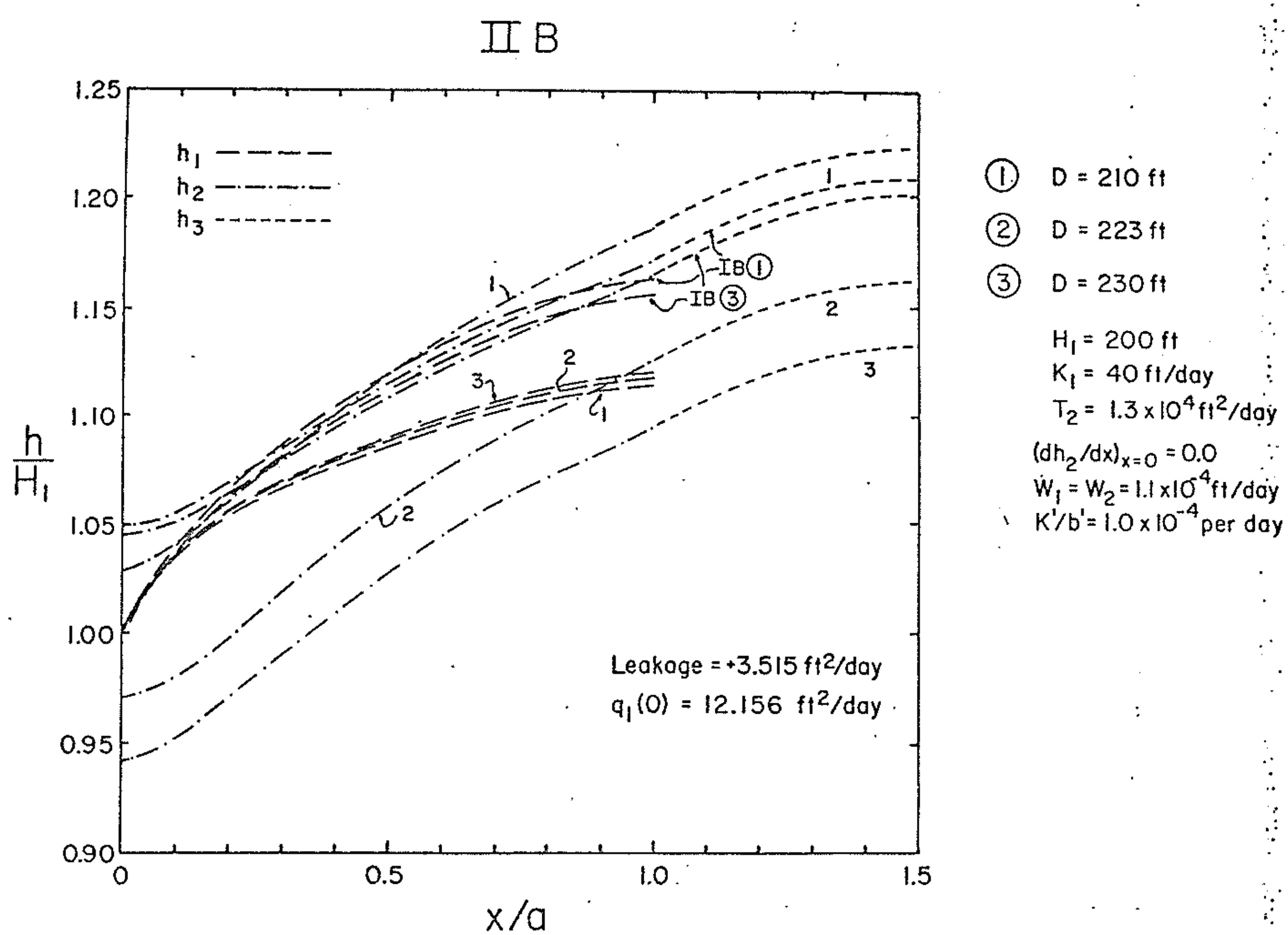

Figure 3-9. Effect of changes in the weighted average depth of the flow profile in the shallow aquifer on the groundwater flow in the three-aquifer system (case II-B) and a comparison of results with case I-B ($W_1=W_2$).

APPENDIX

```
C*****************************************************************************
C
C
C                    THREE-AQUIFER PROBLEM CASE I-A
C
C
C     C       X --------  DISTANCE FROM THE STREAM
C     C       L --------  WIDTH OF THE BASIN
C     C       INX ------  TOTAL NUMBER OF NODES IN X-DIRECTION
C     C       INX1 -----  NUMBER OF NODES WITHIN THE DISTANCE A
C     C       K1,K3 ----  HYDRAULIC CONDUCTIVITIES OF THE SHALLOW AND INTAKE-AREA
C     C                   AQUIFERS, RESPECTIVELY
C     C       A --------  WIDTH OF THE CONFINED AND SHALLOW AQUIFERS
C     C       DK3 ------  INCREMENTAL CHANGE IN K3
C     C       DK1 ------  INCREMENTAL CHANGE IN K1
C     C       W1 -------  RATE OF PERCOLATION IN THE SHALLOW AQUIFER
C     C       W2 -------  RATE OF INFILTRATIONM IN THE INTAKE -AREA AQUIFER
C     C       T1,T2,T3 -  TRANSMISSIVITIES OF THE SHALLOW, CONFINED, INTAKE-AREA
C     C                   AQUIFERS, RESPECTIVELY
C     C       DT2 ------  INCREMETAL CHANGE IN TRANSMISSIVITY OF CONFINED AQUIFER
C     C       KB -------  LEAKAGE COEFFICIENT
C     C       B2 -------  THICKNESS OF  CONFEINED AQUIFER
C     C       DINIT ----  SATURATED THICKNESS OF THE SHALLOW AQUIFER
C     C       H1 -------  HYDRAULIC HEAD IN SHALLOW AQUIFER AT THE LOCATION OF THE
C     C       STREAM
C     C       HT -------  HYDRAULIC HEAD IN CONFINED AQUIFER AT THE LOCATION OF THE
C     C                   STREAM
C     C       QIA10 ----  OUTFLOW FROM SHALLOW AQUIFER WHEN X=0, CASEI-A
C     C       QIA20 ----  OUTFLOW FROM CONFINED AQUIFER WHEN X=0,CASE I-A
C     C       Q3 -------  OUTFLOW FROM INTAKE-AREA AQUIFER WHEN X=A
C*****************************************************************************
C
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION HA(30),H2(30),H3(30),H1SQ(30),X(30)
      DIMENSION DD(30)
C
      INX=20
      INX1=10
      INXP1=INX1+1
C
C     READ IN VALUE FOR X IN MILES
C
      PRINT 30
      DO 10 I=1,INX
      READ 45,X(I)
      X(I)=X(I)*5280.0
   10 PRINT 31,I,X(I)
```

```
      DK1=20.0D00
      DT2=10000.D00
      DK3=0.00005
      DK3=20.0D00
      K3=160.0D00
      DO 205 IK3=1,5
      DW1=0.0002
      DW2=0.0002
      W2=0.73D-03
      W1=0.73D-03
      K1=60.0D00
      DO 202 IK1=1,2
      T2=140000.D00
      DO 203 IT2=1,5
      KB=0.00001
      DO 204 IKB=1,2
      B2=300.0D00
      H1=85.0000
      HT=160.00
      A=9.0*5280.0
      L=30.0*5280.0
      ASQ=A*A
      DINIT=150.0D00
      D=DINIT
      D=150.0D00
      LA=L-A
      T1=K1*D
      T3=K3*B2
      H12=H1*H1
      ALPHA1=W1/T1
      ALPHA2=W2/T3
      BE1SQ=T1/KB
      BE2SQ=T2/KB
      BE2=BE2SQ
      BE1=BE1SQ
      DELTA=BE2/BE1
      DLPL1=(1.00D00+DELTA)
      DP1SQ=DLPL1*DLPL1
      BETASQ=(1.00D00+DELTA)/BE2
      BETA=DSQRT(BETASQ)
      BETA4P=BETASQ*BETASQ
      BET1SQ=K1/(2.0D00*KB)
      GAMMA=BE2/BET1SQ
      MUSQ= (1.00D00+GAMMA/D)/BE2
      MU=DSQRT(MUSQ)
      WKM2= 2.0D00*W1/(K1*MUSQ)
      WAK2=2.0D00*W1*A/K1
C
```

```
C       COMPUTE TRIG FUNCTIONS
        SBA=DSINH(BETA*A)
        CBA=DCOSH(BETA*A)
        COTBA=CBA/SBA
        CMUA=DCOSH(MU*A)
        SMUA=DSINH(MU*A)
        TBA=DTANH(BETA*A)
        TMUA=DTANH(MU*A)
        COTMUA=CMUA/SMUA
C
        PRINT 62,HT,D
        PRINT 63,K1,KB,K3
        PRINT 64,T2,T3
        PRINT 65,W1,W2
     62 FORMAT ('1',' H2,    D,     .........', 3(E16.6,5X))
     63 FORMAT (' ',' K1,   KB,    K3.......', 3(E16.6,5X))
     64 FORMAT (' ',' T2,    T3............', 3(E16.6,5X))
     65 FORMAT (' ',' W1,    W2............', 3(E16.6,5X))
        H=100.0D0
        INCLP=1
C
C          COMPUTE I-A
C
        DO 1 I=1,INX1
        SBX=DSINH(BETA*X(I))
        CBX=DCOSH(BETA*X(I))
        SBAX= DSINH(BETA*(A-X(I)))
        CBAX= DCOSH(BETA*(A-X(I)))
        XSQ=X(I)*X(I)
        H3(I)=0
C
C          COMPUTE H1(X)
C
        PART1= (H1+DELTA*HT+(DELTA*ALPHA1/BETASQ))/(BETASQ*BE2SQ)
        PART2= ((ALPHA1*A/BE2SQ+K3*B2*ALPHA2*LA/(T2*BE1SQ))*X(I)/BETASQ)
        PART3= ALPHA1*XSQ/(2*BETASQ*BE2SQ)
        PART4= ((-H1+HT+ALPHA1/BETASQ)*TBA-K3*B2*ALPHA2*LA/(T2*BETA*CBA))*
       $ SBX/(BETASQ*BE1SQ)
        PART5= (-H1+HT+ALPHA1/BETASQ)*CBX/(BETASQ*BE1SQ)
        HA(I)= PART1+PART2-PART3+PART4-PART5
        DO(I)= HA(I)
C
C          COMPUTE H2(X)
C
        TIME1=(H1+DELTA*HT)/(BETASQ*BE2SQ)-ALPHA1/(BETA4P*BE2SQ)
        TIME4= ((-H1+HT+ALPHA1/BETASQ)*TBA-K3*B2*ALPHA2*LA/(T2*BETA*CBA))*
       $ SBX/(BETASQ*BE2SQ)
        TIME5= (-H1+HT+ALPHA1/BETASQ)*CBX/(BETASQ*BE2SQ)
```

```
      1  H2(I)=TIME1+PART2-PART3-TIME4+TIME5
C
C        COMPUTE H3(X)
C
         DO 2 I=INXP1,INX
         HA(I)=0
         H2(I)=0
         TERM1= -ALPHA2*(L-X(I))**2/2+ALPHA2*LA**2/2
         TERM2= ALPHA1*ASQ/(2.0000*BETASQ*BE2SQ)
         TERM3= K3*B2*ALPHA2*LA/(T2*BETASQ*BE2SQ)*(A*DELTA+TBA/BETA)
         TERM4= (-H1+HT+ALPHA1/BETASQ)/(BETASQ*BE2SQ*CBA)
         TERM5= (H1+DELTA*HT)/(BETASQ*BE2SQ)-ALPHA1/(BETA+P*BE2SQ)
       2 H3(I)= TERM1+TERM2+TERM3+TERM4+TERM5
         PRINT 92,H1
      92 FORMAT(2X,'*****************   H1=   ',F10.0)
         PRINT 8
         PRINT 9
         DO 3 I=1,INX
       3 PRINT 4,X(I),HA(I),H2(I),H3(I)
C  ***********************************************
C
C
C
C
C        COMPUTE   QIA10
C
         SES=ALPHA1*A/(BETASQ*BE2SQ)
         BSB=(-H1+HT+ALPHA1/BETASQ)*TBA/(BETA*BE1SQ)
         SEO=K3*B2*ALPHA2*LA/(T2*BE1SQ*BETASQ)
         UH=1-1/CBA
         QIA10=SES+BSB+SEO*UH
C
C
C        COMPUTE   QIA20
C
         SES=ALPHA1*A/(BETASQ*BE2SQ)
         BSB=(-H1+HT+ALPHA1/BETASQ)*TBA/(BETA*BE2SQ)
         SEU=K3*B2*ALPHA2*LA/(T2*BE2SQ*BETASQ)
         ECB=DELTA+1/CBA
         QIA20= SES-BSB+SEU*ECB
C
         PRINT 75,QIA10
         PRINT 76,QIA20
         PRINT 79,Q3A
         KB=KB+DKB
     204 CONTINUE
         T2=T2+DT2
     203 CONTINUE
```

```
0127          K1=K1+DK1
0128      202 CONTINUE
0129      206 CONTINUE
0130      205 CONTINUE
0131          K3=K3+DK3
0132      208 CONTINUE
0133        4 FORMAT(21X,4E20.7)
     C
0134        8 FORMAT ('-',46X,'*** THAQUIFER *** PART I-A ***'/)
0135        9 FORMAT (30X,'X=',15X,' H1(X)=', 14X,'H2(X1=',14X,'H3(X)='/)
0136       30 FORMAT ('1',39X,'NODE',12X,'X'/)
0137       31 FORMAT (41X,I3,2X,E20.7)
0138       45 FORMAT (F3.0)
0139       75 FORMAT ('-',10X,' THE VALUE OF  QIA10 IS.......',G20.5)
0140       76 FORMAT ('-',10X,' THE VALUE OF  QIA20 IS.......',G20.5)
0141       79 FORMAT('-',10X,'THE VALUE OF Q3 IS',G20.5)
     C
     C
     C
0142          STOP
0143          END
```

NODE	X
1	0.0
2	0.52800000D+04
3	0.10560000D+05
4	0.15840000D+05
5	0.21120000D+05
6	0.26400000D+05
7	0.31680000D+05
8	0.36960000D+05
9	0.42240000D+05
10	0.47520000D+05
11	0.47520000D+05
12	0.52800000D+05
13	0.63360000D+05
14	0.73920000D+05
15	0.84480000D+05
16	0.95040000D+05
17	0.10560000D+06
18	0.12672000D+06
19	0.14784000D+06
20	0.15840000D+06

H2, D, ...	0.1600000D+03	0.1500000D+03	
K1, KB, K3.......	0.6000000D+02	0.1000000D-04	0.1600000D+03
T2, T3............	0.1400000D+06	0.4800000D+05	
W1, W2............	0.7300000D-03	0.7300000D-03	
**************** H1=	85.		

*** THAQUIFER *** PART 1-A ***

X=	H1(X)=	H2(X)=	H3(X)=
0.0	0.8500000D+02	0.1600000D+03	0.0
0.5280000D+04	0.1085398D+03	0.1627750D+03	0.0
0.1056000D+05	0.1281279D+03	0.1656587D+03	0.0
0.1584000D+05	0.1442832D+03	0.1666177D+03	0.0
0.2112000D+05	0.1574155D+03	0.1716257D+03	0.0
0.2640000D+05	0.1678393D+03	0.1746624D+03	0.0
0.3168000D+05	0.1757841D+03	0.1777132D+03	0.0
0.3696000D+05	0.1814019D+03	0.1807681D+03	0.0
0.4224000D+05	0.1847723D+03	0.1838222D+03	0.0
0.4752000D+05	0.1859052D+03	0.1868747D+03	0.0
0.4752000D+05	0.0	0.0	0.1868747D+03
0.5280000D+05	0.0	0.0	0.1955664D+03
0.6336000D+05	0.0	0.0	0.2116778D+03
0.7392000D+05	0.0	0.0	0.2260952D+03
0.8448000D+05	0.0	0.0	0.2386128D+03
0.9504000D+05	0.0	0.0	0.2498363D+03
0.1056000D+06	0.0	0.0	0.2591640D+03
0.1267200D+06	0.0	0.0	0.2727315D+03
0.1478400D+06	0.0	0.0	0.279[illegible]52D+03
0.1584000D+06	0.0	0.0	0.2805532D+03

THE VALUE OF QIA10 IS....... .48711D-02

THE VALUE OF QIA20 IS....... .51280D-03

THE VALUE OF Q3 IS .98013D+06

```
C**************************************************************************
C
C
C
C                     THREE-AQUIFER PROBLEM, CASES I-B AND II-B
C
C
C**************************************************************************
C
C
      IMPLICIT REAL*8 (A-H,J-Z)
      DIMENSION DD(30)
      DIMENSION HA(30),H2(30),H3(30),H1SQ(30),X(30)
      INX=20
      INX1=10
      INXP1=INX1+1
C
C     READ IN VALUE FOR X IN MILES
C
      PRINT 30
      DO 10 I=1,INX
      READ 45,X(I)
      X(I)=X(I)*5280.0
   10 CONTINUE
C
      DO 13 I=1,7
      IK=I+8
   13 PRINT 60,I,X(I),IK,X(IK)
      PRINT 61,X(8)
C
C
C
      W2=0.10D-05
      QONE=1.0E-04
      KB=1.5E-04
      K1=88.0000
      T2=137000.D00
      D=60.0
      DO 204 ID=1,5
      W1=0.10D-05
      DO 203 IW=1,3
      W2=W1
      DINIT=0
      DO 202 IMPC=1,3
      IF(IMPC.GT.1) QONE=0.0D00
      IF(IMPC.GT.2) QONE=1.0E-05
      B2=300.0D00
      K3=33.33D00
```

```
0032          H1=85.0D00
0033          HT=160.D0
0034          A=9.0*5280.0
0035          L=30.0*5280.0
0036          H12=H1*H1
0037          ASQ=A*A
0038          LA=L-A
0039          QTWO=QONE
0040          T1=K1*D
0041          T3=K3*B2
0042          ALPHA1=W1/T1
0043          ALPHA2=W2/T3
0044          BE1SQ=T1/KB
0045          BE2SQ=T2/KB
0046          BE2=BE2SQ
0047          BE1=BE1SQ
0048          INX=15
0049          DELTA=BE2/BE1
0050          DLPL1=(1.00D00+DELTA)
0051          DP1SQ=DLPL1*DLPL1
0052          BETASQ=(1.00D00+DELTA)/BE2
0053          BETA=DSQRT(BETASQ)
0054          BETA4P=BETASQ*BETASQ
0055          BET1SQ=K1/(2.00D00*KB)
0056          GAMMA=BE2/BET1SQ
0057          MUSQ= (1.00D00+GAMMA/D)/BE2
0058          MU=DSQRT(MUSQ)
0059          WKM2= 2.0D00*W1/(K1*MUSQ)
0060          SMUA=DSINH(MU*A)
0061          CMUA=DCOSH(MU*A)
0062          TMUA=DTANH(MU*A)
0063          COTMUA=CMUA/SMUA
0064          WAK2=2.00D0*W1*A/K1
        C
        C     COMPUTE TRIG FUNCTIONS
        C
0065          SBA=DSINH(BETA*A)
0066          CBA=DCOSH(BETA*A)
0067          COTBA=CBA/SBA
0068          TBA=DTANH(BETA*A)
0069          PRINT 62,HT,D,QONE
0070          PRINT 63,K1,KB,K3
0071          PRINT 64,T2,T3
0072          PRINT 65,W1,W2
0073          H=100.0D00
0074          INCLP=1
        C
        C        COMPUTE I-B
```

```
                C
0075                  DO 1 I=1,INXL
0076                  SBX=DSINH(BETA*X(I))
0077                  CBX=DCOSH(BETA*X(I))
0078                  SBAX= DSINH(BETA*(A-X(I)))
0079                  CBAX= DCOSH(BETA*(A-X(I)))
0080                  XSQ=X(I)*X(I)
0081                  H3(I)=0
                C
                C        COMPUTE H1(X)
                C
0082                  MULTA= K3*B2*ALPHA2*LA*DELTA/(T2*BETA*DLPL1)
0083                  PIECEA= (DELTA+CBA)*COTBA-SBA
0084                  MULTB= ALPHA1*A*DELTA/DLPL1
0085                  PIECEB= (COTBA/BETA-BE2SQ/(A*DLPL1))
0086                  PIECEC=  -QONE*DELTA*COTBA/BETA
0087                  PIECED= BE2SQ*ALPHA1*DELTA/DP1SQ
0088                  PIECEE= (ALPHA1*A+MULTA*BETA*DLPL1)*X(I)/DLPL1
0089                  PIECEF= -ALPHA1*XSQ/(2*DLPL1)
0090                  PIECEG= MULTA*SBAX
0091                  MULTH= DELTA/DLPL1*CBAX
0092                  PIECEH= -(ALPHA1*A/(BETA*SBA)+MULTA*DLPL1/( SBA*DELTA)*(DELTA+CBA)
                     $   -QONE*DLPL1/(BETA*SBA))
0093                  HA(I)= H1+MULTA*PIECEA+MULTB*PIECEB+PIECEC+PIECED+PIECEE+PIECEF
                     $   +PIECEG+MULTH*PIECEH
0094                  DD(I)=HA(I)
                C
                C        COMPUTE H2(X)
                C
0095                  LINEA= PIECEA
0096                  LINEB= PIECEB
0097                  LINEC= PIECEC
0098                  LINED=-PIECED/DELTA
0099                  LINEE= PIECEE
0100                  LINEF= PIECEF
0101                  LINEG=-PIECEG/DELTA
0102                  LINEH=-PIECEH
0103                  MULTH2=MULTH/DELTA
0104              1   H2(I)= H1+MULTA*LINEA+MULTB*LINEB+LINEC+LINED+LINEE+LINEF+LINEG
                     $  +MULTH2*LINEH
                C
                C        COMPUTE H3(X)
                C
0105                  BITA= (A*BETA+(DELTA+CBA)*COTBA-SBA+(DELTA+CBA)/(DELTA*SBA))
0106                  MULTB3= ALPHA1*A/DLPL1
0107                  BITB= (DELTA/BETA*COTBA-BE2SQ/A+A/2+1/(BETA*SBA))
0108                  BITC= -QONE/BETA*(DELTA*COTBA+1/SBA)
0109                  BITD=  ALPHA2*LA*LA/2
```

```
      DO 2 I=INXP1,INX
      H2(I)=0
      HA(I)=0
      BITE=-ALPHA2*(L-X(I))*(L-X(I))/2
    2 H3(I)=  H1+MULTA*BITA+MULTB3*BITB+BITC+BITD+BITE
      PRINT 8
      PRINT 9
      DO 3 I=1,INX
    3 PRINT 4, X(I),HA(I),H2(I),H3(I)
C     ****************************************************************
C
C
C
C
C        COMPUTE II-B
      MULNW= K3*B2*W2*LA*GAMMA/(T2*T3)
C
C        COMPUTE H3(X)
C
      MULTBB= 1/(MUSQ*BE2SQ*D)
      MULNG=MULNW/GAMMA
      BITB=   MULNG/MU*(GAMMA*A*MU+GAMMA*(GAMMA/D+CMUA)*COTMUA-GAMMA*
     $SMUA+D/SMUA*(GAMMA/D+CMUA))
      BITC= WAK2*(1/(MU*SMUA)-1/(A*MUSQ)+A/2+GAMMA/(D*MU)*
     $  (COTMUA-1/(A*MU)))
      BITD= H12*MUSQ*BE2SQ
      BITE= -QTWO*MU* BE2SQ/SMUA*(GAMMA*CMUA+D)
      BITF= W2*LA*LA/(2*T3)
      DO 21 I=INXP1,INX
      HISQ(I) =0
      H2(I)=0
      BITA= -W2*(L-X(I))*(L-X(I))/(2*T3)
   21 H3(I)= BITA+(BITB+BITC+BITD+BITE)*MULTBB+BITF
      DO 20 I=1,INX1
      CMUX=DCOSH(MU*X(I))
      SMUX=DSINH(MU*X(I))
      CMUAX=DCOSH(MU*(A-X(I)))
      SMUAX=DSINH(MU*(A-X(I)))
      XSQ=X(I)*X(I)
C
C
C        COMPUTE H1 (X)**2
C
      H3(I)=0
      MULNW= K3*B2*W2*LA*GAMMA/(T2*T3)
      PIECEA= -W1*XSQ/K1+(WAK2+MULNW)*X(I)
      PIECEB= WKM2*GAMMA/D+H12*MUSQ*BE2SQ-QTWO*MU*BE2SQ*GAMMA*COTMUA
      PIECEC= WAK2*GAMMA/(D*MU)*(COTMUA-1/(A*MU))
```

```
0143          PIECED= MULNW/MU*((GAMMA/D+CMUA)*COTMUA-SMUA)
0144          PIECEE= MULNW/MU*SMUAX
0145          PIECEF= (QTWO*MJ*BE2SQ/SMUA-WAK2/(D*MU*SMUA)-MULNW/(MU*SMUA*GAMMA)
             $  *(GAMMA/D+CMUA))
0146          MULTF= GAMMA*CMUAX
0147          HISQ(I)= (PIECEA+PIECEB+PIECEC+PIECED+PIECEE+PIECEF*MULTF)/
             $       (MUSQ*BE2SQ)
0148          IF(HISQ(I).LT.0.0)  HISQ(I)=0.0
0149          HA(I)=DSQRT(HISQ(I))
        C
        C        COMPUTE H2(X)
        C
0150          LINEA= PIECEA
0151          LINEB= PIECEB   -WKM2*GAMMA/D
0152          LINEC= PIECEC
0153          LINED= PIECED    -WKM2
0154          LINEE=-PIECEE   *D/GAMMA
0155          LINEF=-PIECEF
0156          MULTF2= D*CMUAX
0157       20 H2(I)= (LINEA+LINEB+LINEC+LINED+LINEE+LINEF*MULTF2)/(MUSQ*BE2SQ*D)
0158          PRINT 22
0159          PRINT 23
0160          DO 24 I=1,INX
0161       24 PRINT 4,X(I),HA(I),H2(I),H3(I)
0162       91 CONTINUE
        C
        C
        C
        C
        C
        C
        C
        C
        C
        C        COMPUTE QIB10
0163          MICHEL=ALPHA1*A
0164          JOE=K3*B2*ALPHA2*LA*DELTA/T2 -QONE*DELTA
0165          QIB10=(MICHEL+JOE)*T1
        C        COMPUTE QIIB10
0166          JOEL= 2*W1*A/K1
0167          JOB=K3*B2*W2*LA*GAMMA/(T2*T3)
0168          QIIB10=(JOEL+JOB)*K1/2.0D00
        C
0169          PRINT 70, QIB10
0170          PRINT 71,QIIB10
        C
0171      202 CONTINUE
0172          W1=W1*5.0D00
```

```
      203 CONTINUE
          D=D+20.0
      204 CONTINUE
        4 FORMAT (21X,4E20.7)
        8 FORMAT ('-',46X,'*** THAQUIFER *** PART I-B ***'/)
        9 FORMAT (30X,'X=',15X,' H1(X)=', 14X,'H2(X)=',14X,'H3(X)='/)
       22 FORMAT ('-',46X,'*** THAQUIFER *** PART II-B  **'/)
       23 FORMAT (30X,'X=',15X,' H1(X)=', 14X,'H2(X)=',14X,'H3(X)='/)
       30 FORMAT('1',46X,'VALUES OF X',/)
       45 FORMAT (F3.0)
       60 FORMAT(20X,I3,2X,F13.2,15X,I3,2X,F13.2)
       61 FORMAT (20X,'  B',2X,F13.2)
       62 FORMAT ('1',' H2,    D,    QONE.....', 3(E16.6,5X))
       63 FORMAT (' ',' K1,   KB,    K3.......', 3(E16.6,5X))
       64 FORMAT (' ',' T2,    T3............', 3(E16.6,5X))
       65 FORMAT (' ',' W1,    W2............', 3(E16.6,5X))
       70 FORMAT ('-',10X,' THE VALUE OF  QIB10 IS......',G20.5)
       71 FORMAT ('-',10X,' THE VALUE OF QIIB10 IS......',G20.5)
C
C
C
          STOP
          END
```

VALUES OF X

1	0.0	9	42240.00
2	5280.00	10	47520.00
3	10560.00	11	47520.00
4	15840.00	12	52800.00
5	21120.00	13	63360.00
6	26400.00	14	73920.00
7	31680.00	15	84480.00
8	36960.00		

H2, D, QUNE......	0.1600000D+05	0.6000000D+02	0.1000000D-03
K1, KB, K3........	0.8800000D+02	0.1500000D-03	0.3333000D+02
T2, T3.............	0.1870000D+06	0.9999000D+04	
W1, W2..............	0.1000000D-05	0.1000000D-05	

*** THAQUIFER *** PART I-B ***

X=	H1(X)=	H2(X)=	H3(X)=
0.0	0.8500000D+02	0.6366190D+02	0.0
0.5280000D+04	0.7278829D+02	0.6421110D+02	0.0
0.1056000D+05	0.6783778D+02	0.6435512D+02	0.0
0.1584000D+05	0.6583223D+02	0.6441585D+02	0.0
0.2112000D+05	0.6502105D+02	0.6444271D+02	0.0
0.2640000D+05	0.6469422D+02	0.6445574D+02	0.0
0.3168000D+05	0.6456381D+02	0.6446307D+02	0.0
0.3696000D+05	0.6451310D+02	0.6446801D+02	0.0
0.4224000D+05	0.6449497D+02	0.6447187D+02	0.0
0.4752000D+05	0.6449076D+02	0.6447520D+02	0.0
0.4752000D+05	0.0	0.0	0.6447520D+02
0.5280000D+05	0.0	0.0	0.6453235D+02
0.6336000D+05	0.0	0.0	0.6463830D+02
0.7392000D+05	0.0	0.0	0.6473310D+02
0.8448000D+05	0.0	0.0	0.6481674D+02

*** THAQUIFER *** PART II-B **

X=	H1(X)=	H2(X)=	H3(X)=
0.0	0.8500000D+02	0.9073477D+02	0.0
0.5280000D+04	0.7722606D+02	0.9103591D+02	0.0
0.1056000D+05	0.7489301D+02	0.9112366D+02	0.0
0.1584000D+05	0.7422405D+02	0.9115123D+02	0.0
0.2112000D+05	0.7403573D+02	0.9116175D+02	0.0
0.2640000D+05	0.7398369D+02	0.9116737D+02	0.0
0.3168000D+05	0.7397005D+02	0.9117149D+02	0.0
0.3696000D+05	0.7396715D+02	0.9117510D+02	0.0
0.4224000D+05	0.7396713D+02	0.9117845D+02	0.0
0.4752000D+05	0.7396746D+02	0.9118165D+02	0.0
0.4752000D+05	0.0	0.0	0.9118165D+02
0.5280000D+05	0.0	0.0	0.9123880D+02
0.6336000D+05	0.0	0.0	0.9134475D+02
0.7392000D+05	0.0	0.0	0.9143955D+02
0.8448000D+05	0.0	0.0	0.9152319D+02

THE VALUE OF QIBIO IS...... -18.542