# PRELIMINARY EVALUATION OF PROFESSOR C. E. JACOB'S CONTRIBUTIONS IN THE FIELD OF WATER RESOURCES OF NEW MEXICO

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## ABSTRACT

From 1965 till his death on January 30, 1970, Professor C. E. Jacob was chairman of Ground Water Hydrology at New Mexico Institute of Mining and Technology. During the last three decades ground water studies have advanced considerably due to his efforts, original achievements, and philosophies. He is therefore considered by many as one of the founders of modern groundwater hydrology. In this study, taken from unpublished reports and notes, Jacob projects the decline of the water table in the Ogallala aquifer of the Southern High Plains of New Mexico and Texas. After developing approximate analytical solutions for steady and unsteady flow as a function of flow depth, slope, water table configuration, and recharge, numerical results are obtained by digital computation. Representative values of porosity and hydraulic conductivity, required for this purpose, were obtained from an analysis of well data. The author concludes that the hypothetical time required for draining the aquifer, without pumping and without recharge, is from 5,000 to 8,000 years. At pumping rates prevailing during the decade 1951-1960, with natural recharge and discharge operating, the aquifer will be depleted in less than 100 years. Starting with complete exhaustion, the aquifer would recover to within 2 percent of its initial (1938) storage in 5,600 to 5,700 years. If pumpage had ceased at the end of 1958, recovery to within 2 percent of initial storage would probably not exceed 1,500 years. These estimates assume that no local or temporary climatic fluctuations of the water table take place during the time spans considered.

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# C. E. JACOB'S STUDY ON THE PROSPECTIVE AND HYPOTHETICAL FUTURE IN THE MINING OF THE GROUNDWATER DEPOSITED UNDER THE SOUTHERN HIGH PLAINS OF TEXAS AND NEW MEXICO

by

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## INTRODUCTION

The personal collection of books, writings, and consulting reports of the late Professor C. E. Jacob was recently donated by his heirs to the library of the New Mexico Institute of Mining and Technology because of Professor Jacob's close association with the hydrology department from 1965 until his untimely death in January, 1970.

Stimulated by this donation and encouraged by a grant from the New Mexico Water Resources Research Institute, we began an analysis of C. E. Jacob's collection. The main objective of this analysis was to identify and to publish original contributions to the field of Water Resources of New Mexico. Hydrologists are fully cognizant of the great impact of Professor Jacob's earlier work in the field of groundwater hydrology. However, almost all of his later work is in the form of unpublished consulting reports; it was recognized that much original work and techniques would never become known or made useful to hydrologists unless a study of this nature were undertaken.

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We could identify only one study in this collection with results which are directly applicable to New Mexico water resources. It is in the form of a memorandum report submitted to Mr. W. L. Broadhurst, then Chief Hydrologist of the High Plains Underground Water Conservation District No. 1, Lubbock, Texas, entitled, On the Prospective and Hypothetical Future of the Mining of the Groundwater Deposited Under the Southern High Plains of Texas and New Mexico, dated September 1961. It is concerned with the High Plains aquifer in the Ogallala formation of upper Tertiary age. It was prepared for introduction as testimony in the 1962 suit, "Marvin and Mildred Shurbet vs. The United States of America". This suit, wherein the plaintiffs were claiming an income tax allowance for their capital costs in the groundwater beneath their land that was being depleted to create income, was sponsored by the High Plains Underground Water Conservation District No. 1. Mr. Jacob had been retained by the District as a consulting hydrologist, and as an expert witness.

The District's efforts were ultimately successful, and landowners in the Southern High Plains of Texas and New Mexico who can demonstrate a cost in the groundwater beneath their land are now allowed an income tax credit for the value of the water that is annually depleted from beneath their land to create income.

In New Mexico, the Ogallala aquifer is the most important aquifer in Quay, Curry, Roosevelt, and Lea counties (Figure 1). With continued water level decline and pollution problems, its rational management and conservation are of grave concern at present and have been the objective of several research projects sponsored by the New Mexico Water Resources

Research Institute (NMWRRI).

With the memorandum were found Jacob's original handwritten analysis, method of solution, and calculations to this problem. It was soon apparent that this study was of far reaching consequences. Publication of this analysis would indeed be a significant contribution, as well as a posthumous honor to Professor Jacob. The original report was not written for publication as a technical paper. In order to prepare Jacob's work for publication, a considerable amount of rewriting had to be done; however, care was taken to conserve, as much as possible, Jacob's original discussion.

Acknowledgment. The High Plains Underground Water Conservation District No. 1 (Lubbock, Texas) and its manager, Frank A. Rayner, granted permission for publication of all of the material Jacob had prepared for that District. Mr. Rayner also supplied Figure 2a and reviewed the manuscript. This review of Jacob's work was supported by the New Mexico Water Resources Research Institute as Project No. 3109-133.

## STATEMENT OF THE PROBLEM

In the original problem statement it was proposed to reduce the actual three-dimensional to a two-dimensional case and analyze it for a strip running from west (New Mexico) to east (Texas). The problem statement was summarized in the form of four questions.

1. Assuming no natural recharge and no pumpage, how long will it take for the entire South Plains Ogallala formation to reach the stage where no well will produce water?

2. Assuming present water levels in the formation, with natural recharge and natural discharge, how long will it take to exhaust the water in the formation (within the meaning of [1] above), assuming pumpage at the rate of:

   a. the 1951 to 1960 average

   b. the maximum year during 1951 - 1960

   c. the minimum year during 1951 - 1960

Calculate curves, assuming 0.15 inches per year (in/yr) for natural recharge.

3. Assuming the exhaustion point indicated in [2] above has occurred and assuming no subsequent pumpage, how long will it take for

the entire formation to return to its original saturated equilibrium condition?

4. Assuming no pumping of any kind after the end of 1958, but assuming natural recharge (at the rate indicated in [2] above) and natural discharge, how long would it take to restore the formation to its original natural equilibrium, and at what depth would the static water level be under the Shurbet farm?

## DESCRIPTION OF AREA

Figure 1 is a map of the selected area of the Southern High Plains water study. The scale is approximately one inch to 20.5 miles. The area is about 16,900 square miles or 10,800,000 acres. Also shown in Figure 1 is the trace of a composite section passing through the Shurbet well (Fig. 2a). Its slope is about ten feet per mile. The hydraulics of the aquifer will be studied by considering an idealized strip paralleling that section.

## HYDROLOGIC CHARACTERISTICS

About 250 million acre-feet (maf) of water storage were originally present, of which 40 maf have been taken out. The groundwater in that area discharges along a scarp which marks the eastern boundary of the Ogallala. The recharge appears to average about 0.15 in/yr. The average porosity being used in this study is about 15 percent and the hydraulic conductivity is about 400 gallons per day per square foot ($gpd/ft^2$).

Much of the validity of this study hinges upon the choice of the

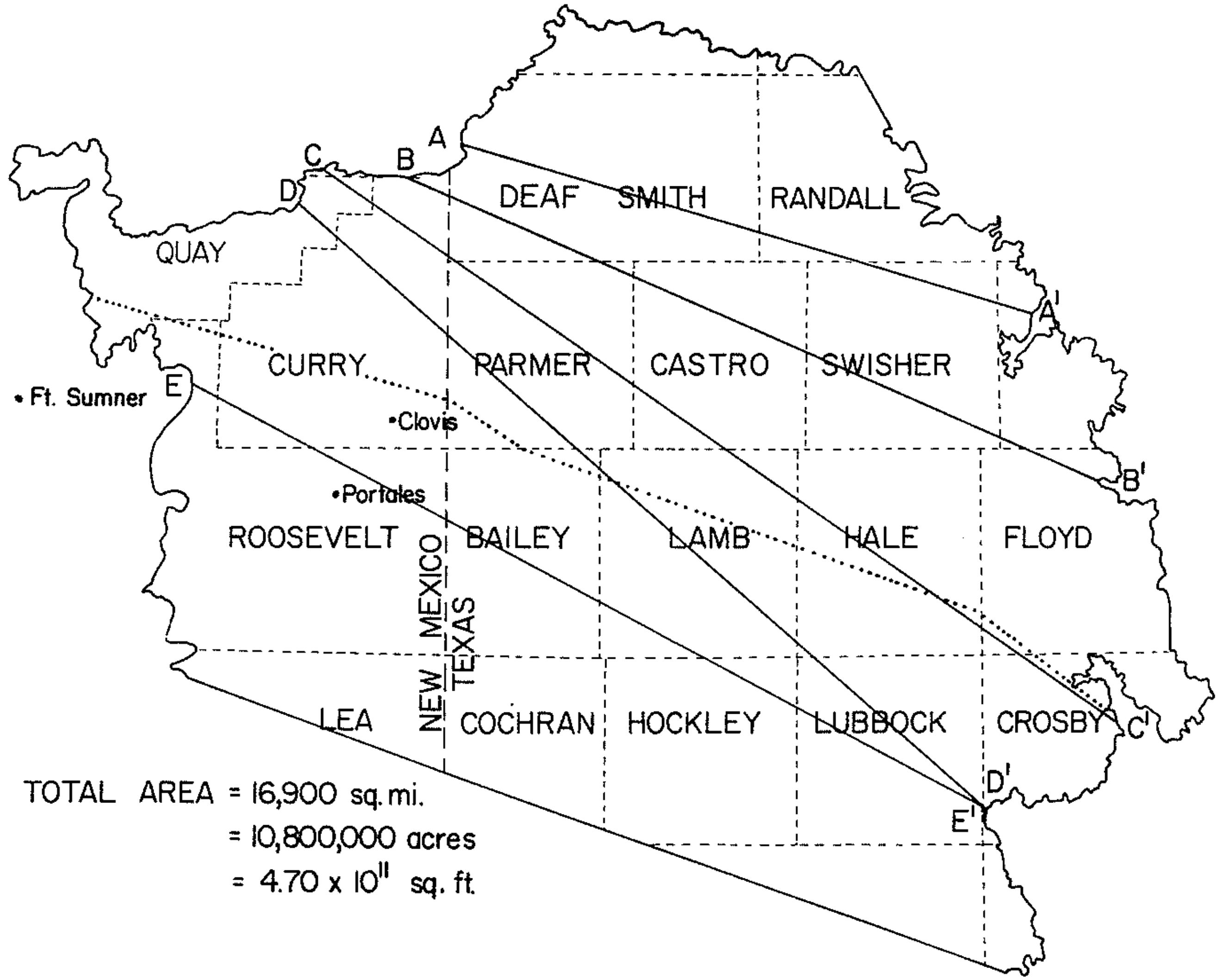


Figure 1: Map of part of Southern High Plains. Dotted trace: composite section through Shurbet farm shown in Fig. 2a. AA', BB', CC', DD', EE' are traces of sections shown in Fig. 2b-2c.

natural recharge at 0.15 in/yr. The approximate calculation of this parameter and its implications are discussed below.

Figure 2 shows profiles of the aquifer bottom and water table configuration (1938) along the sections shown in Figure 1.

## MATHEMATICAL ANALYSIS

### Steady-state Analysis

The purpose of the steady-state analysis is to determine initial depth of flow or hydrologic equilibrium conditions for a finite strip, assuming unidirectional flow.

The general non-steady, one-dimensonal flow equation for a water table aquifer with recharge is (Fig. 3).

$$\frac{\partial}{\partial x'} Ky \frac{\partial y'}{\partial x'} = S \frac{\partial y'}{\partial t} - w \qquad (1)$$

wherein y = depth of flow

y' = the point elevation of the water table with respect to an arbitrary horizontal datum

$\left|\frac{\partial y'}{\partial x'}\right|$ = hydraulic gradient

K = hydraulic conductivity

w = uniform recharge from all sources (natural and return flow)

and S = storage coefficient.

Boundary conditions are

$$y = 0 \text{ at } x' = 0 \qquad (1b)$$

(the depth of the discharge at the scarp is negligible), and

$$y = 0, \frac{y\partial y'}{\partial x'} = 0 \text{ at } x' = x_o' \qquad (1c)$$

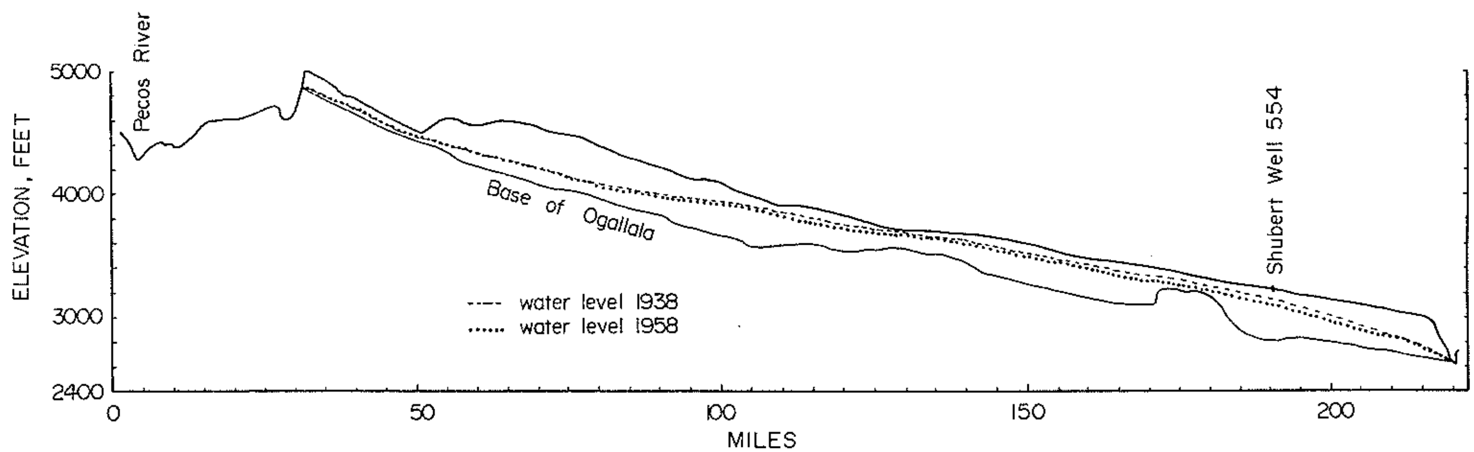


Figure 2: Hydrologic sections through the Southern High Plains.
(a) Composite section through the Shurbet farm. Dotted trace on Fig. 1.

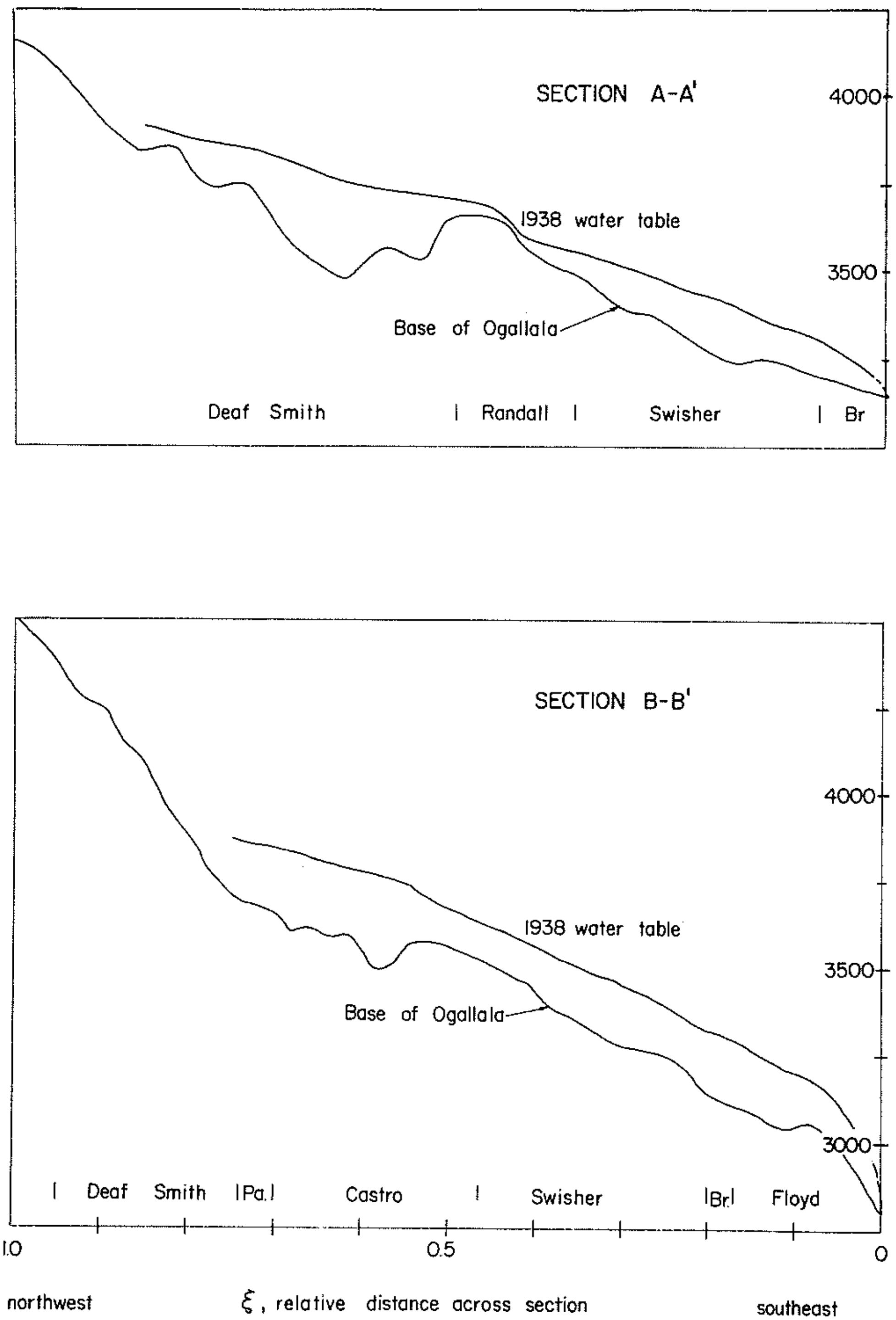


Figure 2: Hydrologic sections through the Southern High Plains.
(b) Saturated thickness of Ogallala formation, corresponding to 1938, along sections A-A' and B-B' of Fig. 1.

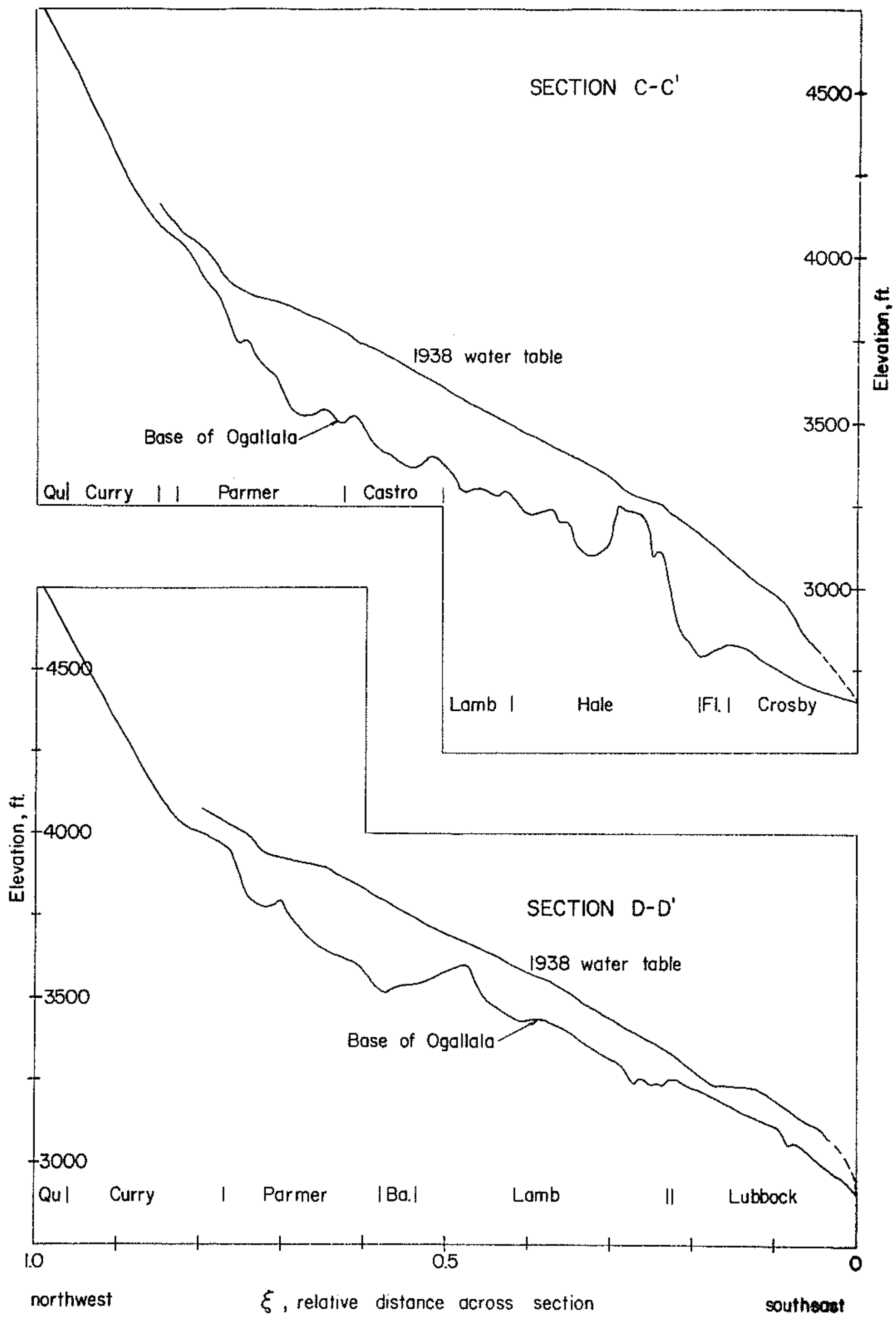


Figure 2: Hydrologic sections through the Southern High Plains.
(c) Saturated thickness of Ogallala formation, corresponding to 1938, along sections C-C' and D-D' of Fig. 1.

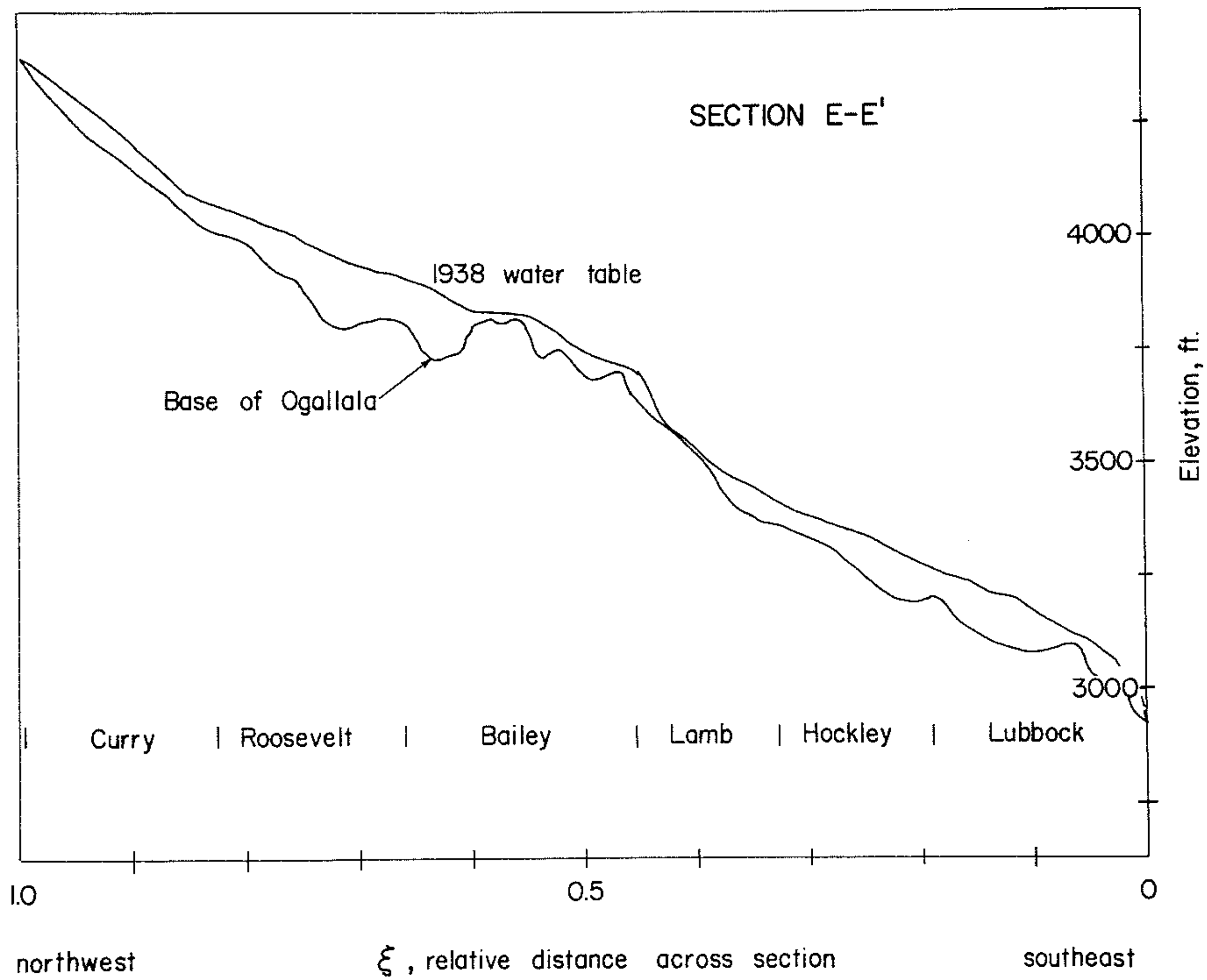


Figure 2: Hydrologic sections through the Southern High Plains. (d) Saturated thickness of Ogallala formation, corresponding to 1938, along section E-E' of Fig. 1.

where $x'_o$ is the boundary of the recharge zone, at which the depth and flow are zero.

For the gently sloping aquifer shown in Figure 3,

$$y \simeq y' - x'\alpha \tag{2}$$

With these substitutions, Eq. (1) becomes, for the steady state,

$$\frac{d}{dx'}\left[(y' - x'\alpha)\frac{dy'}{dx'}\right] = -\frac{w}{K} \tag{3a}$$

$$y = y' = 0 \text{ at } x' = 0 \tag{3b}$$

$$y = 0,\ (y' - x'\alpha)\frac{dy'}{dx'} = 0 \text{ at } x' = x'_o \tag{3c}$$

Equation (3a) can be integrated directly with respect to $x'$ from $x'$ to $x'_o$,

$$\left[(y' - x'\alpha)\frac{dy'}{dx'}\right]_{(x'_o,\ y'_o)} - (y' - x'\alpha)\frac{dy'}{dx'} =$$

$$-\frac{w}{K}\int_{x'}^{x'_o} dx' = -\frac{w}{K}(x'_o - x') \tag{4a}$$

Equation (4a) is simplified using Eq. (3c).

$$(y' - x'\alpha)\frac{dy'}{dx'} = \frac{w}{K}(x'_o - x') \tag{4b}$$

The boundary condition (3b) is retained.

$$y' = 0 \text{ at } x' = 0$$

Set $(w/K)^{\frac{1}{2}}(x' - x'_o) = x$ (4c)

and from (2):

$$\frac{dy'}{dx'} = \frac{dy}{dx'} + \alpha \tag{4d}$$

With these substitutions, Eq. (4b) can be expressed in terms of the flow depth (2) as:

$$y\frac{dy}{dx} + ay + x = 0 \tag{5a}$$

where $a = \frac{\alpha}{(w/K)^{\frac{1}{2}}}$ is the relative slope, a non-dimensional number.

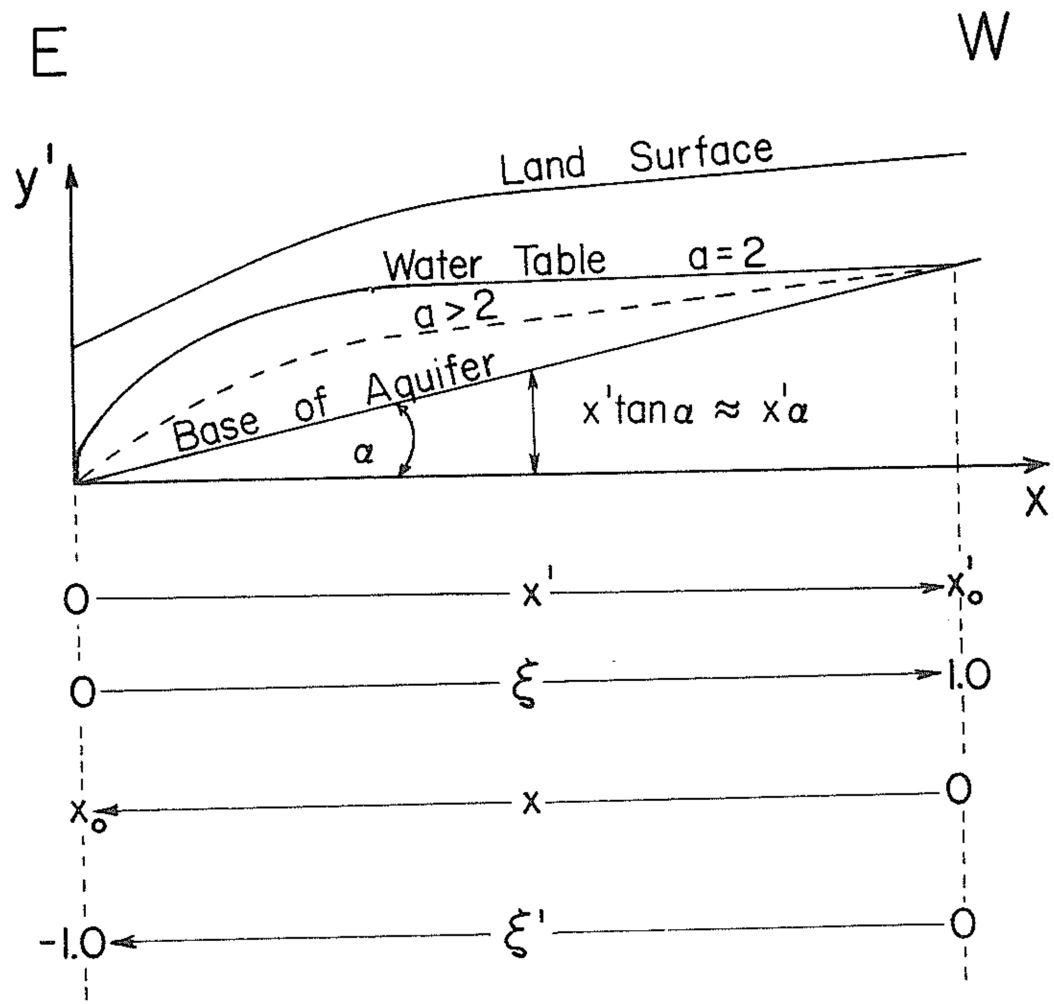


Figure 3: Schematic flow profile and coordinate systems used in the mathematical analysis. The east and west ends of this profile are reversed in order to make it conform with the mathematical curves given in subsequent figures. An alternative would have been to reverse the curves (the majority of the figures) and to change coordinate systems and notations from those used by the author originally.

The boundary condition (3b) becomes

$$y = 0 \text{ at } x_o = -(w/K)^{\frac{1}{2}}\, x_o' = -\, \alpha/a\ (x_o') \tag{5b}$$

The solution of Eq. (5a) cannot be obtained in explicit form. There are three implicit forms of the solution, depending on whether a is equal to, greater than, or less than 2. *)

$$(x \pm y) \exp\left[\frac{x}{x \pm y}\right] = C \text{ for } a = \pm 2 \tag{6a}$$

$$C_1(y - \alpha x)^{\alpha} = C_2(y - \beta x)^{\beta} \text{ for } |a| > 2 \tag{6b}$$

wherein $|C_1| + |C_2| > 0$ and $\alpha, \beta = -\frac{1}{2} a \pm \frac{1}{2}\sqrt{a^2 - 4}$

$$\ln (y^2 + axy + x^2) - \frac{2a}{\sqrt{4 - a^2}} \tan^{-1} \frac{2y + ax}{x\sqrt{4 - a^2}} = C \tag{6c}$$

for $|a| < 2$.

Jacob did some study on all solution possibilities. However, only the solutions with $a \geq 2$ (small w/K) are pertinent to the problem at hand.

## Non-dimensional Form

By expressing the variables as well as the constants in non-dimensional form, solutions of Eq. (5a) can be obtained in the form of universal curves.

Let $\xi' = \dfrac{x' - x_o'}{x_o'}$ = dimensionless horizontal distance reckoned from the upper (west) end of the aquifer (see Fig. 3). (7a)

$\eta = \dfrac{y}{x_o'\sqrt{w/K}}$ = dimensionless flow depth (7b)

*) Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Chelsea Publishing Co., New York, 1948 (p. 329).

$$\tau = \frac{t\sqrt{w\ K}}{Sx_o'} = \text{dimensionless time} \tag{7c}$$

$$\upsilon = \int \eta d\xi' = \frac{\int y dx}{x_o'^2 \sqrt{w/K}} \tag{7d}$$

$$= \text{dimensionless aquifer volume.}$$

The following numerical values were used in the dimensionless transformation:

$$x_o' = 10^6 \text{ feet}$$

$$x_o'(w/K)^{\frac{1}{2}} = 800 \text{ feet}$$

$$\frac{Sx_o'}{(wK)^{\frac{1}{2}}} = 10^4 \text{ years}$$

$$x_o'^2(w/K)^{\frac{1}{2}} = 8 \times 10^8 \text{ cubic feet per foot.}$$

How these values were obtained is discussed in the section on Hydrologic Data below.

Transformed into dimensionless form, Eq. (5a) becomes

$$\eta\frac{d\eta}{d\xi'} + a\eta + \xi' = 0 \tag{8}$$

For a <2 and using the dimensionless variables, the solution is:

$$\ln\ (\xi'^2 + a\xi'\eta + \eta^2) = \frac{2a}{\sqrt{4-a^2}}\left[\tan^{-1}\frac{a\xi' + 2\eta}{\xi'\sqrt{4-a^2}} - \tan^{-1}\frac{a}{\sqrt{4-a^2}}\right] \tag{9}$$

For a = 2 the solution is:

$$\xi' + \eta = -\exp\left[\eta/(\xi' + \eta)\right] \tag{10}$$

For a >2

$$\left(\frac{\eta-\alpha\xi'}{\alpha}\right)^{\alpha} = \left(\frac{\eta-\beta\xi'}{\beta}\right)^{\beta} \tag{11}$$

where $\alpha, \beta = \dfrac{-a \pm \sqrt{a^2 - 4}}{2}$

The boundary condition for $a \geq 2$ is

$$\eta = 0, \; \xi' = -1 \tag{12}$$

The head is zero at the left-hand boundary (see Fig. 3). Rigorously, this left-hand boundary is a seepage face, and mathematically very complex. In this study, the author was concerned with the regional hydrological characteristics rather than with local conditions in the immediate vicinity of the boundary. Therefore Eq. (12) is appropriate.

The mathematical solutions (10) and (11) for $a \geq 2$ place $\eta'_{max}$, the groundwater divide, at the right-hand (western) boundary. Therefore, there is no flow across the right-hand (western) boundary; all of the outflow is through the left-hand boundary (the scarp at the eastern boundary of the Ogallala outcrop). This is in accordance with the physical characteristics of the problem. Continuity therefore requires that

$$\eta = 0, \; \xi' = 0, \text{ and} \tag{12a}$$

$$\eta \frac{d\eta}{d\xi'} = 1, \; \xi' = -1 \tag{12b}$$

The boundary condition for $a < 2$ is

$$\eta = 0, \; \xi' = \xi'_o = \exp\left[\frac{-\pi a}{\sqrt{4-a^2}}\right]$$

Thus $\eta \neq 0$ at the right-hand boundary (see curve for $a = 1$ in Fig. 4). This is inconsistent with the boundary condition (3c) which was used in developing subsequent equations. There is outflow through the right-hand boundary which requires the existence of a groundwater divide within the flow region. An examination of the profiles in Fig. 2 shows that this condition does not actually exist. Jacob himself did not use this solution in his analysis. It is included because it could have relevance to some physical situation. In general, situations giving rise to a < 2 in a chosen domain are not dependent only on conditions within that domain. Appropriate coupling conditions to an adjacent domain must be provided at the boundary and solved as a multi-domain problem.

Solutions are shown graphically in Fig. 4. Note that, since $\underline{a} = \alpha\sqrt{K/w}$, they can be used to estimate natural recharge rates by assuming steady-state profiles, provided a good average value of $\underline{K}$ is given. A value of $\underline{a}$ is determined by finding the best fit of the field profile with calculated profiles. For this study the 1938 "steady-state" was utilized.

Transient Analysis

This section deals with the decay and recovery of water table profiles, assuming unidirectional flow over a sloping bed, and using different combinations of natural recharge and pumping. This analysis leads to answers for Questions 1 through 4 of the problem statement.

The differential equation for unsteady flow with steady recharge at a point along the strip is:

$$K\frac{\partial}{\partial x'}\left[(y' - \alpha x')\frac{\partial y'}{\partial x'}\right] = S\frac{\partial y'}{\partial t} - w \qquad (13)$$

The boundary conditions for the unsteady state and a $\geq$ 2 are

$$y = y' - \alpha x' = 0, \; x' = 0, \text{ and } x' = x'_o \qquad (13a)$$

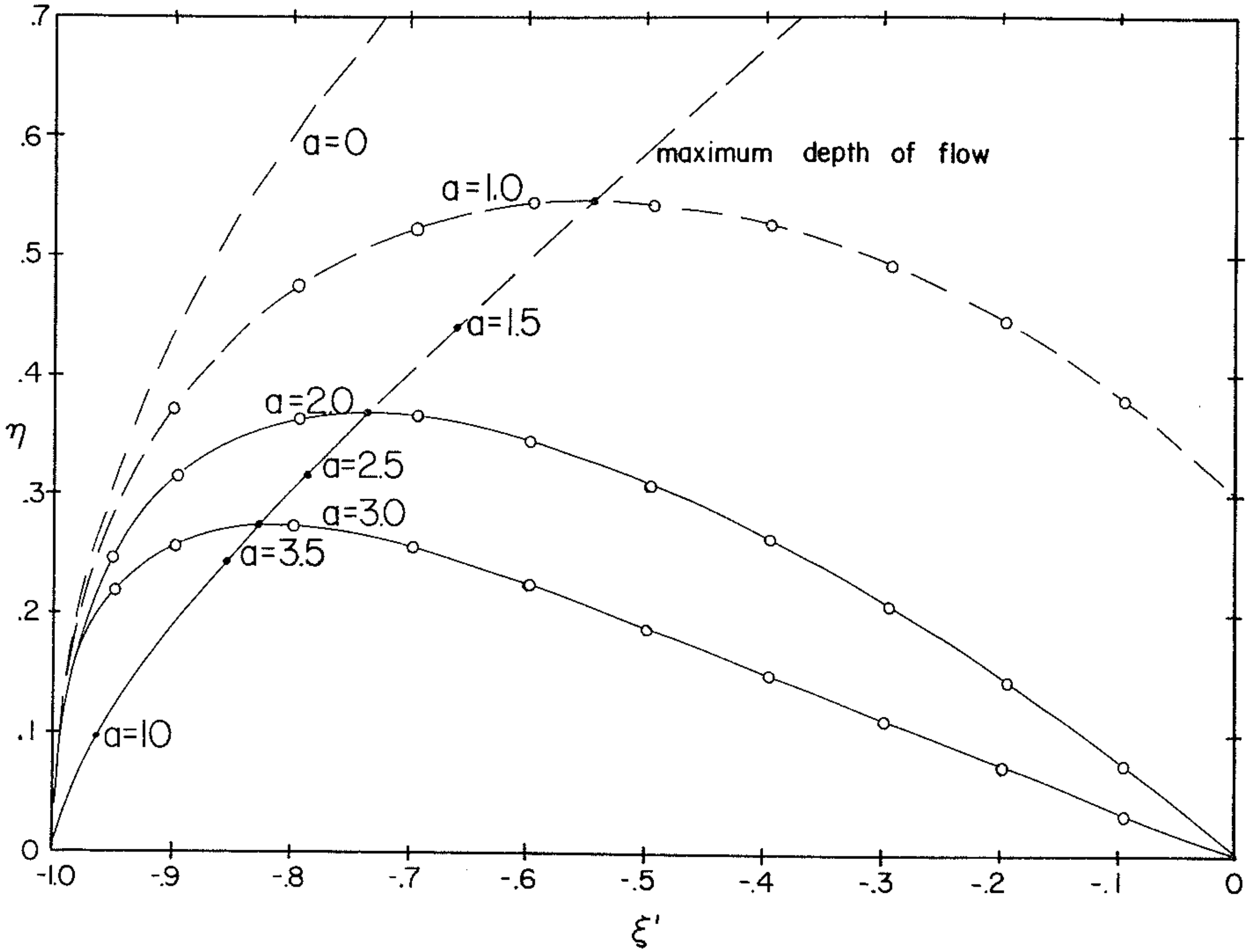


Figure 4: Solutions of differential equation for depth of unidirectional ground-water flow over inclined plane bottom with uniform recharge.

The continuity condition is

$$y\frac{dy}{dx'} = 0, \; x' = x_o' \qquad (13b)$$

The steady-state solutions, Eqs. (6a-c) are obtained from Eq.(13) by setting $S\frac{\partial y'}{\partial t} = 0$.

Next, the unsteady-state equation in terms of the non-dimensional variables defined in Eqs. (7a-d) is derived:

$$K\frac{\partial}{\partial x'}\left[y\left(\frac{\partial y}{\partial x'} + \alpha\right)\right] = S\frac{\partial y}{\partial t} - w \qquad (14)$$

Dividing through by w,

$$\frac{1}{(w/K)}\,\frac{\partial}{\partial x'}\left[y\left(\frac{\partial y}{\partial x'} + \alpha\right)\right] = \frac{S}{w}\,\frac{\partial y}{\partial t} - 1 \qquad (14a)$$

$$\partial x' = x_o'\,\partial\left(\frac{x'}{x_o'}\right) \qquad (14b)$$

Hence:

$$\frac{1}{(w/K)}\,\frac{\partial}{x_o'\partial\left(\frac{x'}{x_o'}\right)}\left[y\left(\frac{\partial y}{x_o'\partial\left(\frac{x'}{x_o'}\right)} + \alpha\right)\right] = \frac{S}{w}\,\frac{\partial y}{\partial t} - 1 \qquad (14c)$$

$$\frac{\partial}{\partial\left(\frac{x'}{x_o'}\right)}\left[\frac{y}{\sqrt{w/K}\;x_o'}\left(\frac{\partial\left[\frac{y}{\sqrt{w/K}\;x_o'}\right]}{\partial\left(\frac{x'}{x_o'}\right)} + \frac{\alpha}{\sqrt{w/K}}\right)\right] = \frac{S}{w}\,\frac{\partial y}{\partial t} - 1 \qquad (15)$$

Further, from Eqs. (7b) and (7c)

$$y = \eta x_o'\sqrt{w/K} \text{ and } t = \tau\frac{Sx_o'}{\sqrt{w\;K}} \qquad (15a)$$

and, from Eqs. (4c) and (7a),

$$\partial\left(\frac{x'}{x_o'}\right) = \partial\left(\frac{x}{x_o}\right) = \partial\xi' \qquad (15b)$$

With these substitutions, (15) becomes

$$\frac{\partial}{\partial\xi'}\left[\eta\left(\frac{\partial\eta}{\partial\xi'} + a\right)\right] = \frac{\partial\eta}{\partial\tau} - 1$$

or:

$$\frac{\partial}{\partial\xi'}\left[\frac{\tfrac{1}{2}\partial(\eta^2)}{\partial\xi'} + a\eta\right] = \frac{\partial\eta}{\partial\tau} - 1$$

$$\frac{\tfrac{1}{2}\partial^2(\eta^2)}{\partial\xi'^2} + a\frac{\partial\eta}{\partial\xi'} = \frac{\partial\eta}{\partial\tau} - 1 \qquad (16)$$

With pumping and natural recharge going on simultaneously, (16) becomes:

$$\frac{\tfrac{1}{2}\partial^2(\eta^2)}{\partial\xi'^2} + a\frac{\partial\eta}{\partial\xi'} = \frac{\partial\eta}{\partial\tau} - \bar{w} \qquad (16a)$$

where $\bar{w} = \frac{w + w'}{w}$ is the dimensionaless net rate of recharge, with w' as the pumping rate, a negative quantity. We can shift the origin of coordinates to the left-hand side by setting

$$\xi = \xi' + 1$$

Then, since $d\xi = d\xi'$, (16a) may be written for a = 2 as

$$\frac{1}{2}\frac{\partial^2(\eta^2)}{\partial\xi^2} + 2\frac{\partial\eta}{\partial\xi} = \frac{\partial\eta}{\partial\tau} - \bar{w} \qquad (17)$$

The following finite differences algorithm was utilized to solve Eq. (17).

$$\eta_{j+1} = \eta_j + \Delta\tau\left[\frac{\Delta^2(\eta^2)}{2\Delta\xi^2} + 2\frac{\Delta\eta}{\Delta\xi} + \bar{w}\right] \qquad (18)$$

where $\eta_j$ and $\eta_{j+1}$ are dimensionless water levels at time level j (known) and at time level j+1 (to be solved for), respectively. The time step $\Delta\tau$ selected to solve Question 2 was 0.0005 (equivalent to five-year time increments) and the starting profile was the 1958 profile.

Since this is an explicit finite difference solution, convergence is assured only when a convergence criterion is satisfied. For this problem, this criterion is:

$$\frac{\Delta\tau}{\Delta\xi^2} \leq \frac{1}{2} \tag{19}$$

which, for $\xi = 0.1$ and $\Delta\tau = 0.0005$, is well within the limit. The grid used for the numerical and digital computations with the specified boundary conditions is shown in Fig. 5.

## Computation of Changes in Storage

The following analysis is based on Jacob's pencilled notes. The author apparently did not use it for the storage calculations discussed below. These were based on numerical difference equations.

We nonetheless include the analysis because it appears to be of practical value and illustrates Jacob's ability for approximate mathematical analysis of complex hydrological problems.

Changes of volume in storage can be obtained by integrating Eq. (16a). Replacing $\xi'$ by $\xi$ and rearranging, (16a) becomes:

$$\frac{\partial\eta}{\partial\tau} = \frac{\partial}{\partial\xi}\left[\eta\left(\frac{\partial\eta}{\partial\xi} + a\right)\right] + \bar{w} \tag{20}$$

$$\eta = \eta(\xi,\tau)$$

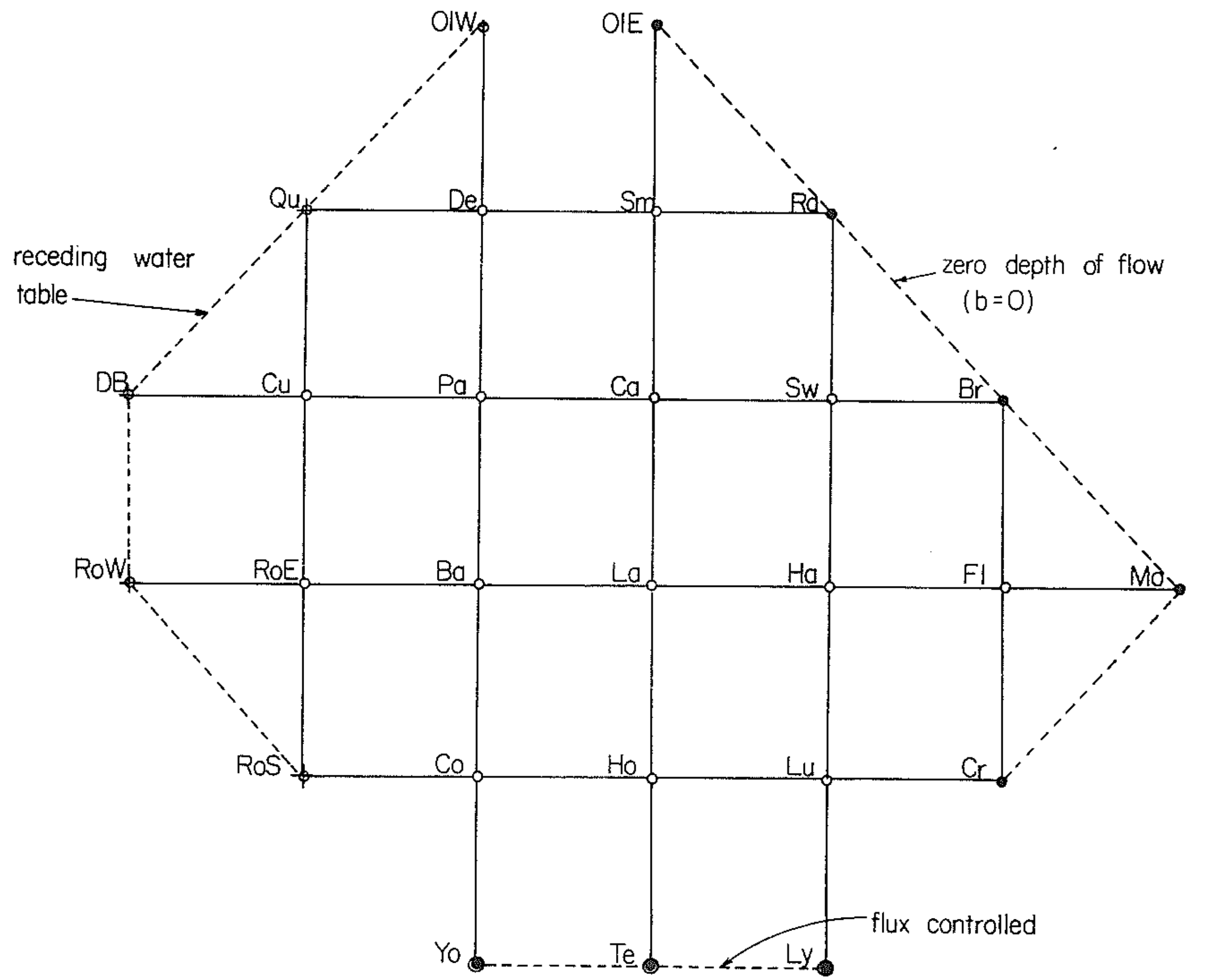


Figure 5: Grid used for digital computations. Circles: 14 internal mesh points at approximate county centers (compare with Figure 1). Dots: 13 boundary mesh points simulating the boundary of the Ogallala aquifer. The grid slightly overreaches the study area.

Boundary conditions are:

$$\eta = 0 \text{ at } \xi = 0; \ \eta = 0 \text{ and } \eta\frac{\partial\eta}{\partial\xi} = 0 \text{ at } \xi = 1 \tag{21}$$

Note that (12b) is not applicable in this unsteady-state case. An initial condition, $\eta(\xi,0)$, also must be specified to define the boundary value problem completely. It is convenient to define the volume variables (see Fig. 6):

$$\upsilon\ (\xi,\tau) = {}_0\!\int^{\xi} \eta(\lambda,\tau)d\lambda \tag{22}$$

$$\bar{\upsilon}\ (\tau) = {}_0\!\int^{1} \upsilon(\xi,\tau)d\xi \tag{23}$$

Expressions for volume in storage can be obtained by integrating (20). Integrating over the interval $[0,\xi]$, and rearranging,

$$\frac{\partial\upsilon}{\partial\tau} = \eta\frac{\partial\eta}{\partial\xi} + a\eta - \left(\frac{\eta\partial\eta}{\partial\xi} + a\eta\right)_{\xi=0} + \xi\bar{w} \tag{24}$$

Applying the boundary conditions (21),

$$\frac{\partial\upsilon(1,\tau)}{\partial\tau} = -\left(\eta\frac{\partial\eta}{\partial\xi}\right)_{\xi=0} + \bar{w} \tag{25}$$

a continuity condition. Then Eq. (24) becomes

$$\frac{\partial\upsilon(\xi,\tau)}{\partial\tau} = \eta\frac{\partial\eta}{\partial\xi} + a\eta + \frac{\partial\upsilon(1,\tau)}{\partial\tau} - (1-\xi)\bar{w} \tag{26}$$

Integrating Eq. (26) with respect to $\xi$ over $[0,1]$,

$$\frac{\partial\bar{\upsilon}(\tau)}{\partial\tau} = \frac{\eta^2}{2}\Bigg|_0^1 + a\upsilon(1,\tau) + \frac{\partial\upsilon(1,\tau)}{\partial\tau} - \frac{\bar{w}}{2} \tag{27}$$

Applying boundary conditions and rearranging,

$$\frac{\partial\upsilon(1,\tau)}{\partial\tau} - \frac{\partial\bar{\upsilon}(\tau)}{\partial\tau} = -a\upsilon(1,\tau) + \frac{\bar{w}}{2} \tag{28}$$

The values $\upsilon(1,\tau)$ and $\bar{\upsilon}(\tau)$ will tend to vary together. Eq. (28) can be solved approximately by assuming a constant ratio,

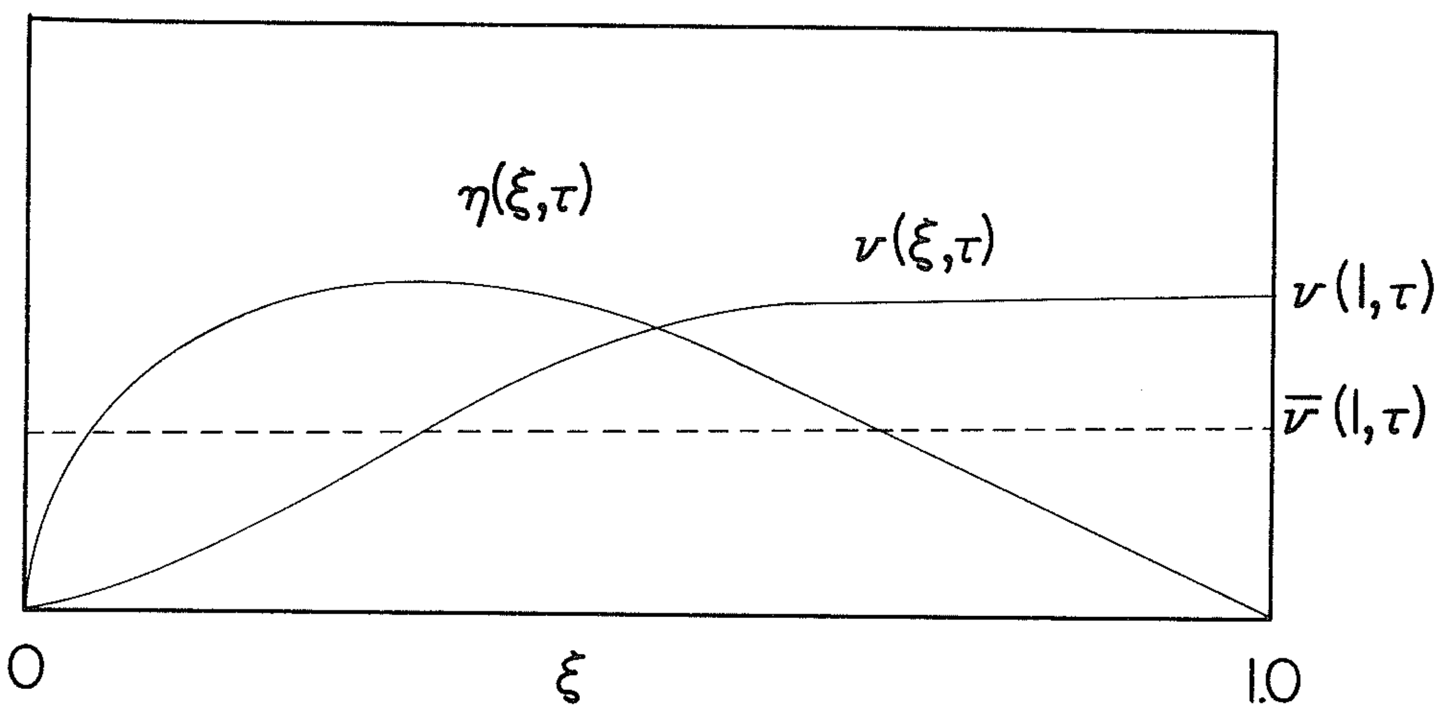


Figure 6: Sketch of dimensionless volume variables and their relation to dimensionless flow depth and distance.

$$b = \frac{\bar{\upsilon}(\tau)}{\upsilon(1,\tau)} \tag{29}$$

For steady recharge or pumping, the ratio will approach a constant and the solution based on constant b will be accurate for long time estimates. It may not be accurate for short times.

Equation (29) is used to eliminate $\bar{\upsilon}(\tau)$ in (28).

$$\frac{\partial \upsilon(1,\tau)}{\partial \tau} = -\frac{a}{1-b}\,\upsilon(1,\tau) + \frac{\bar{w}}{2(1-b)} \tag{30}$$

The solution of Eq. (30) is

$$\upsilon(1,\tau) = \left\{\upsilon(1,0) - \frac{\bar{w}}{(2a)}\right\} e^{-\frac{a\tau}{1-b}} + \frac{\bar{w}}{2a} \tag{31}$$

## HYDROLOGIC DATA

### Hydraulic Conductivity

Seventy-five point determinations of hydraulic conductivity*) yielded an average hydraulic conductivity of 630 gpd/$ft^2$. An average transmissivity of 69,200 gpd/ft was derived from 135 point determinations.*) Dividing the average transmissivity by the average initial depth of flow, namely 200 ft, an average figure of 346 gpd/$ft^2$, is obtained for the hydraulic conductivity. As this assumed initial depth of flow exceeds the average saturated thickness penetrated by the wells, the true average hydraulic conductivity undoubtedly is somewhat higher than indicated by this calculation. On the other hand the hydraulic

*) Credited by Jacob to a report: Pumping Test Data - 1961 - Southern High Plains, Texas and New Mexico, by R.A. Scalapino of William F. Guyton and Associates.

conductivity calculated using the permeable material shown on the drillers' logs or the saturated thickness of all the material penetrated by an average well is probably too high. Accordingly it is concluded that the average hydraulic conductivity probably lies somewhere between 350 and 630 gpd/ft$^2$.

## Specific Yield

By comparing the amount of water pumped from the given areas with the decline of the water table it has been estimated by Broadhurst and others that the average specific yield of the aquifer under the Southern High Plains is about 15 percent. This study uses 15.6 percent in order to make the transformations listed in Eqs. (7) above come out in convenient whole numbers.

## Average Recharge Rate

Estimates made by C. V. Theis (Trans. Am. Geophys. Union 18, 564-568, 1937) for the recharge in Portales Valley and Lea County, New Mexico, are probably too high, with one exception. That is, these estimates are undoubtedly too high for the strip of Southern High Plains under consideration, even though they may be of the correct magnitude for the particular local areas. A better way of getting at the ratio of the average recharge rate to the hydraulic conductivity is by studying the shape of the water table profile.

Referring to the paper by C. V. Theis quoted above, the ratio of the average recharge rate to the hydraulic conductivity can best be estimated from the profiles exhibited in his Figure 3. He says (p. 567) that, from laboratory measurements, "The coefficient of permeability of the coarser, bedded sediments of the Ogallala formation ranged between 80 and 1100 and averaged about 450". He took the average permeability to be 1000 $gpd/ft^2$ "in order to eliminate any chance of underestimation". The values of the relative slopes for the three curves that are shown on his Fig. 3 are as follows:

| | | |
|---|---|---|
| w = 1 in/yr | ½ in/yr | ¼ in/yr |
| a = 1.53 | 2.16 | 3.06 |

Comparing the observed depths of flow with the depths of flow calculated for a = 2.16, Theis found that at four places the observed depths of flow were on the average 90.2 percent of the calculated depths of flow. As the non-dimensional relative slope varies with the inverse of the square root of the recharge rate, it also varies with the inverse of the average relative depth of flow (Eq. 7b). Accordingly the value of relative slope (a) fitting his four points most closely would be about 2.16/0.902 = 2.40.

Using the average hydraulic conductivity cited by Theis, namely 450 $gpd/ft^2$, one calculates the average recharge rate to be 0.183 in/yr. (This is to be compared with the value of 0.150 in/yr assumed in the present study.)

## Ratio of Formation Constants

Actually it is not necessary to know the absolute value of the three coefficients separately, but rather their ratios in pairs. On the basis of available information it would appear that the most probable value of $\sqrt{w/K}$ is 1/1250 = 0.8 x $10^{-3}$. Similarly it would appear that the most probable value of K/S is 125,000 ft/yr. For the present, these values of the ratios will be used. Perhaps it may be found desirable to revise them later after studying the flow in three dimensions using the suggested plane grids.

## Average Slope of Bottom of Aquifer

Theis assumed that the effective average slope of the bottom of the aquifer as seen in a section running from west to east across the Southern High Plains was about 0.002, or roughly 10 ft/mi. Because of the concavity of the bottom of the aquifer on a regional scale, it is now thought best to employ a value of slope that would give results most nearly representing the concave aquifer as regards its time-outflow characteristics. Accordingly an average slope was taken that is somewhat smaller than 0.002, namely 0.0016, or 8.45 ft/mi. (This gives a rise of 1600 feet over the 190-mile stretch of the west-east section.)

Actually using the data first suggested to the author, (C.E. Jacob), namely 400 gpd/$ft^2$ for $\underline{K}$, 0.15 in/yr for $\underline{w}$, and 10 ft/mi for the slope, one would get 2.4 for the value of the relative slope, $\underline{a}$. Dropping down to the smaller average slope, namely 0.0016, one gets 2.0 for the relative slope. This value is to be preferred because an idealized strip with this average constant slope of the bottom would reproduce more closely the outflow characteristics of the actual concave aquifer with uneven bottom.

The slope of the bottom of the concave aquifer at the lower end is actually given by $\underline{a}$ = 1.5, compared to $\underline{a}$ = 2.0 for the plane sloping bottom of the highly idealized strip. (See the two profiles on Fig. 7).

## COMPUTATIONS

### Solutions of Differential Equation for Initial Steady States of Flow

Figure 4 depicts in graphical form the solutions of the fundamental differential equations for four different values of the relative slope, $\underline{a}$, namely 0, 1, 2, and 3.

Eq. (8) is the fundamental differential equation governing the unidirectional flow of groundwater over finite strips with inclined bottom and with uniform recharge. Note that there is only one parameter in this equation, namely $\underline{a}$, the relative slope. Other parameters involving the ratio of the recharge to the hydraulic conductivity or the ratio of the specific yield to the hydraulic conductivity enter the solutions through the boundary conditions or the initial conditions.

The points plotted as circles on Fig. 4 were calculated by repeated trial on the respective implicit equations. The points plotted as dots on Fig. 4 represent the maximum depths of flow for the several values of $\underline{a}$,

Note that the part of the profile shown for $\underline{a}$ = 0 is an arc of a circle whose center is at (0,0).

Unless overshadowed by inhomogeneities and variations in the configuration of the floor of the aquifer, it should be possible from the shape of the water table to estimate the value of $\underline{a}$, which involves the

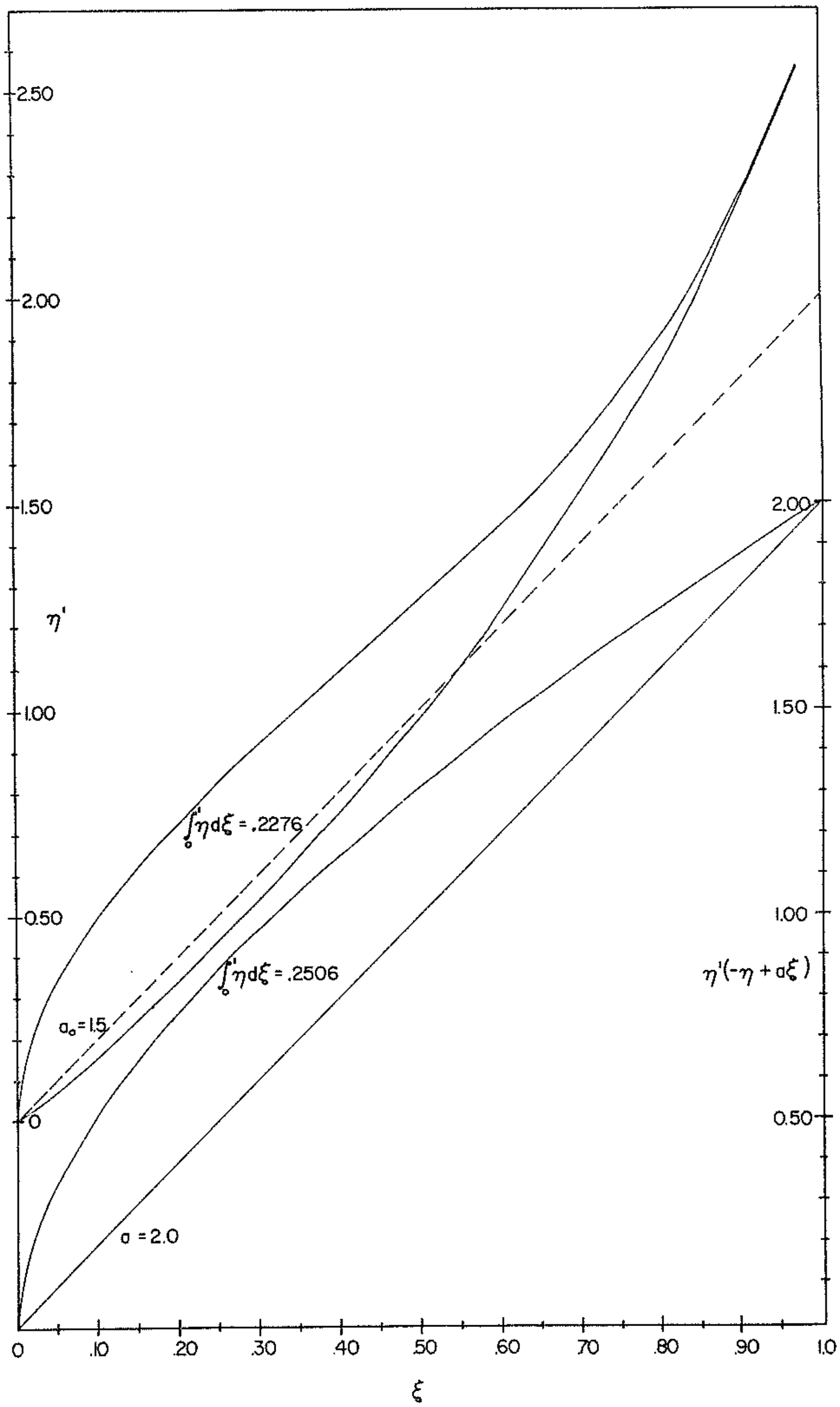


Figure 7: Graphical solutions of differential equation for unidirectional groundwater flow over plane and concave bottoms with uniform recharge.

ratio of the recharge rate to the hydraulic conductivity. For example, if there is no groundwater divide on the profile and if the concavity of the bottom of the aquifer is low, it is certain that the value of a is greater than or at least equal to 2.0. If the position of the maximum depth of flow can also be estimated, then the value of a may be narrowed down somewhat more closely.

Influence of the Concavity of the Bottom of the Aquifer

In order to estimate the influence of the overall concavity of the bottom of the aquifer, a smooth curve was drawn through points representing the average elevation of the bottom of the aquifer over each 20th of the length of the profile. This is shown in the upper of the two profiles on Fig. 7. The lower of the two profiles represents the highly idealized strip with a sloping plane bottom. Both cases were solved by numerical integration of the difference equation corresponding to the differential equation for steady flow (Eq. 18). The lower end or nose of each profile was obtained analytically by integration of the differential equation for a constant bottom slope. In the case of the plane-bottom aquifer, that slope was taken to be $a = 2.0$, and in the case of the concave-bottom aquifer, the slope at the left-hand boundary was taken to be $a = 1.5$.

The graphical solution is accomplished by using the continuity relationship in the normalized variables. At any station the depth of flow times the slope of the water table must equal the length of the profile measured from the right boundary. By drawing a number of slope lines it is easy to project the profile begun at the lower end by analytical solution continuously to the upper end. The fact that the closure is nearly perfect in the plane case shows that the method is very accurate. Accordingly we can accept the results of the graphical integration of the concave case with high assurance of its reliability.

The results of the graphical integration on Fig. 7 are transformed to Fig. 8, where the profile is depicted in terms of depth of flow above the plane or concave bottom, as the case may be. These profiles give a feeling for the effect of the concavity of the aquifer on a regional scale. The saturated volume for a plane bottom, which theoretically is exactly ¼, turned out by summation to be 0.2505, whereas that for the concave bottom was 0.2276, or somewhat smaller. Note that the concavity of the bottom of the aquifer has an effect upon the water table such as to raise it at the lower end and to lower it at the upper end in comparison to what it would be without the concavity.

The influence of the regional concavity of the bottom of the aquifer upon the nonsteady decay of the profile is not known. However by choosing the value of relative slope that we have, namely 2.0 instead of 2.4 as formerly used, we perhaps have duplicated the outflow characteristics of the concave aquifer more closely. Our reason for saying this is that the outflow, of course, is controlled by the shape of the lower end of the profile. Thus we have taken the bottom slope somewhat less than the actual average over the entire profile.

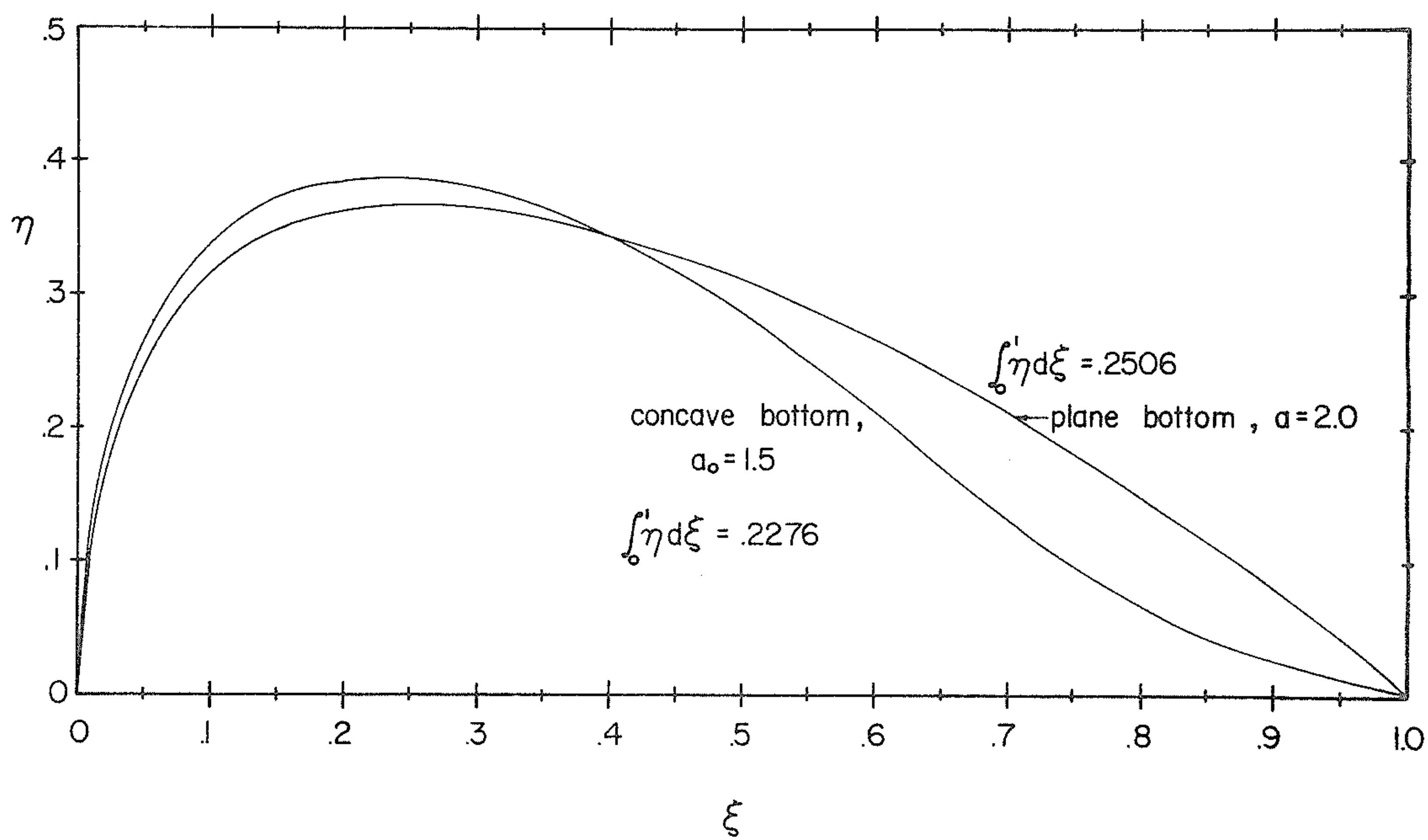


Figure 8: Comparison of groundwater profiles over plane and concave bottoms with uniform recharge.

## Influence of the Unevenness of the Bottom of the Aquifer Upon the Flow Regimen

Figure 9 is a graph comparing the observed and the calculated depth of flow across the Southern High Plains between 1938 and 1958. The curves plotted as full lines give the calculated profiles based upon the highly idealized uniformly sloping strip. The dashed straight line segments give the profile obtained by selecting points at every tenth along the profile from the composite geologic section (Fig. 2a). The unevenness of these dashed segmented lines reflects the unevenness of the bottom of the aquifer. Exactly how this influences the nonsteady state of decay of the profile in the actual case is not known, and will not be known until the problem is solved by numerical methods in three dimensions.*)

The position of Shurbet Well 554 is shown on Fig. 9. Upstream from that well the groundwater flows around a high in the buried Mesozoic landscape. The influence of this and other highs cannot be assessed without analyzing the problem in three dimensions.

*) Such a study is presently in progress through a NMWRRI grant to New Mexico Institute of Mining and Technology. Expected completion date is August 1974. The main objective of this study is to analyze long-term effects of different management practices in New Mexico and Texas.

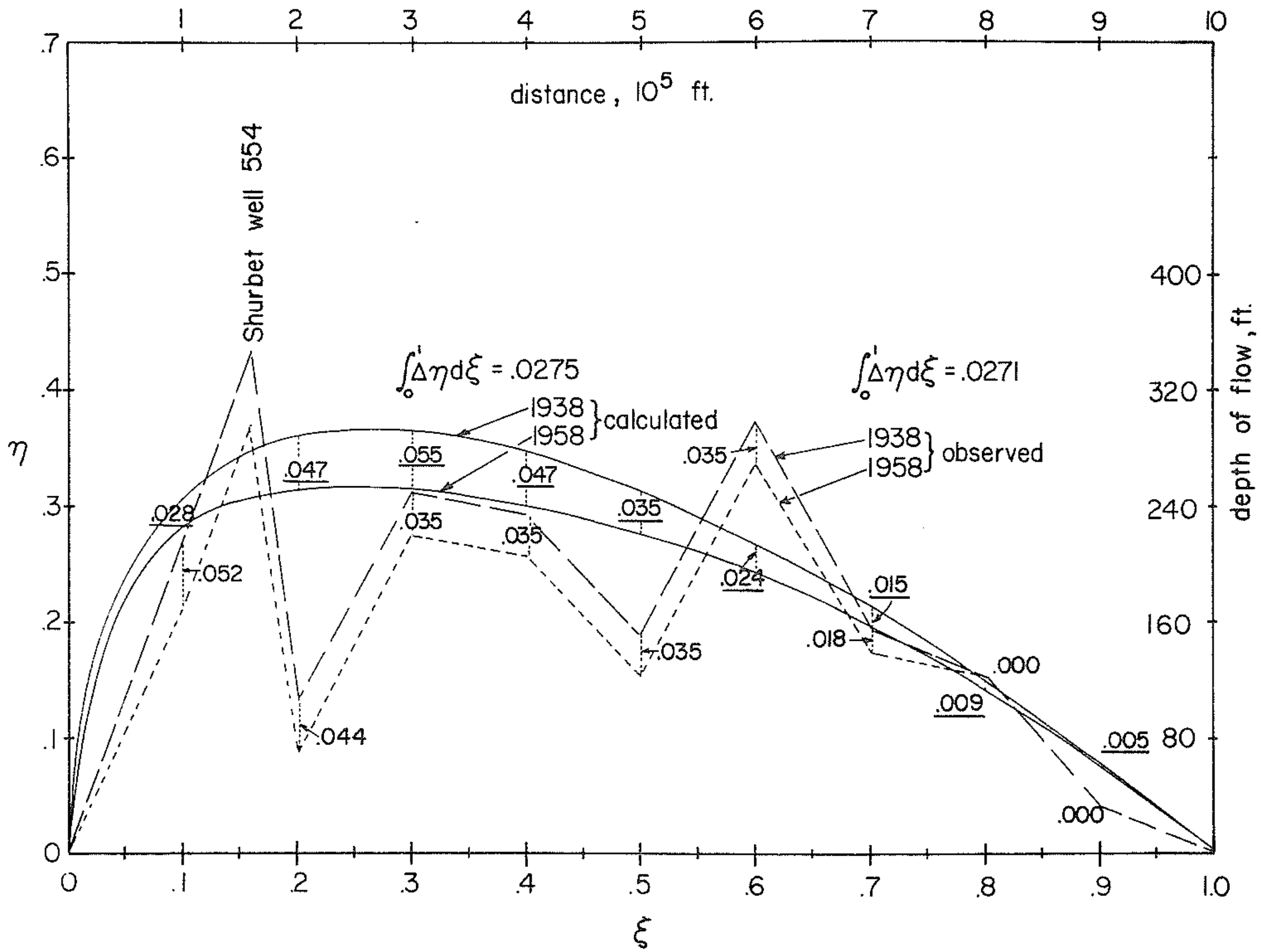


Figure 9: Comparison of observed with calculated reduction of flow depth, 1938 to 1958.

# DISCUSSION OF RESULTS

## Appraisal of Assumed Time Scale

The time scale defined by Eq. (7c) deserves some consideration. Inasmuch as the length of section taken is about the longest that can be drawn in the general direction of groundwater flow from west to east, across the Southern High Plains, namely about 190 miles, or 1,000,000 ft, the times calculated in the analysis which follows are probably maximal. Increasing the value of S would increase the physical time corresponding to each unit of non-dimensional time. Decreasing the value of w or K would have the same effect. The specific yield, as well as the hydraulic conductivity and recharge rate, may be somewhat larger than given above because the percentage of water recycled in the irrigation operation was underestimated. Still, because these factors appear as ratios in the time relationship, their combined net effect on the time scale is probably small. In other words, the possibility of the time scale having been underestimated by reason of underestimating the amount of recirculation is virtually nil.

The influence of the average concavity of the bottom of the aquifer and also the influence of the unevenness of the bottom of the aquifer upon the time constant are not known.

An even more important factor is the shape of the aquifer as seen on the map (Fig. 1). The area of the selected part of the High Plains under investigation is $4.70 \times 10^{11}$ ft$^2$. Therefore the average width of the strip which the selected section represents is 470,000 ft ($14.7 \times 10^{11}/1 \times 10^6$), or about 89 miles. Thus in geometry the selected area more closely resembles an elliptical or rectangular area than a long strip of which the given section might be considered typical. However, it should be pointed out that the thinning out of the aquifer to the southwest along the southwestern boundary of the selected area does constrain the flow somewhat and causes it to occur in a direction parallel to that approximate boundary. The influence of the geometry of the aquifer on a regional scale should be considered in trying to estimate the natural time constant of the aquifer.

In the case of an aquifer of uniform permeability with a flat bottom accepting water at a uniform recharge rate, the fundamental modes of flow are such that the time constant for a circular aquifer is 2.34 times smaller than that for a strip aquifer and the time constant for a square aquifer is 2.00 times smaller than that for a strip aquifer. Whether the contrast is that great in the present case will not be known until the analysis is carried out for the square mesh in three dimensions.

In view of the foregoing it should be stated that estimates of time given in this study are the probable <u>maximum</u> times in light of our present knowledge. By that we mean that the chance that the time involved would exceed that quoted in each instance is probably less than 10 percent.

## Residence Time

Figure 10 shows the time required for an average particle of water to traverse the Ogallala aquifer from northwest to southeast along the more closely resembles an elliptical or rectangular area than a long strip of which the given section might be considered typical. However, it should be pointed out that the thinning out of the aquifer to the southwest along the southwestern boundary of the selected area does constrain the flow somewhat and causes it to occur in a direction parallel to that approximate boundary. The influence of the geometry of the aquifer on a regional scale should be considered in trying to estimate the natural time constant of the aquifer.

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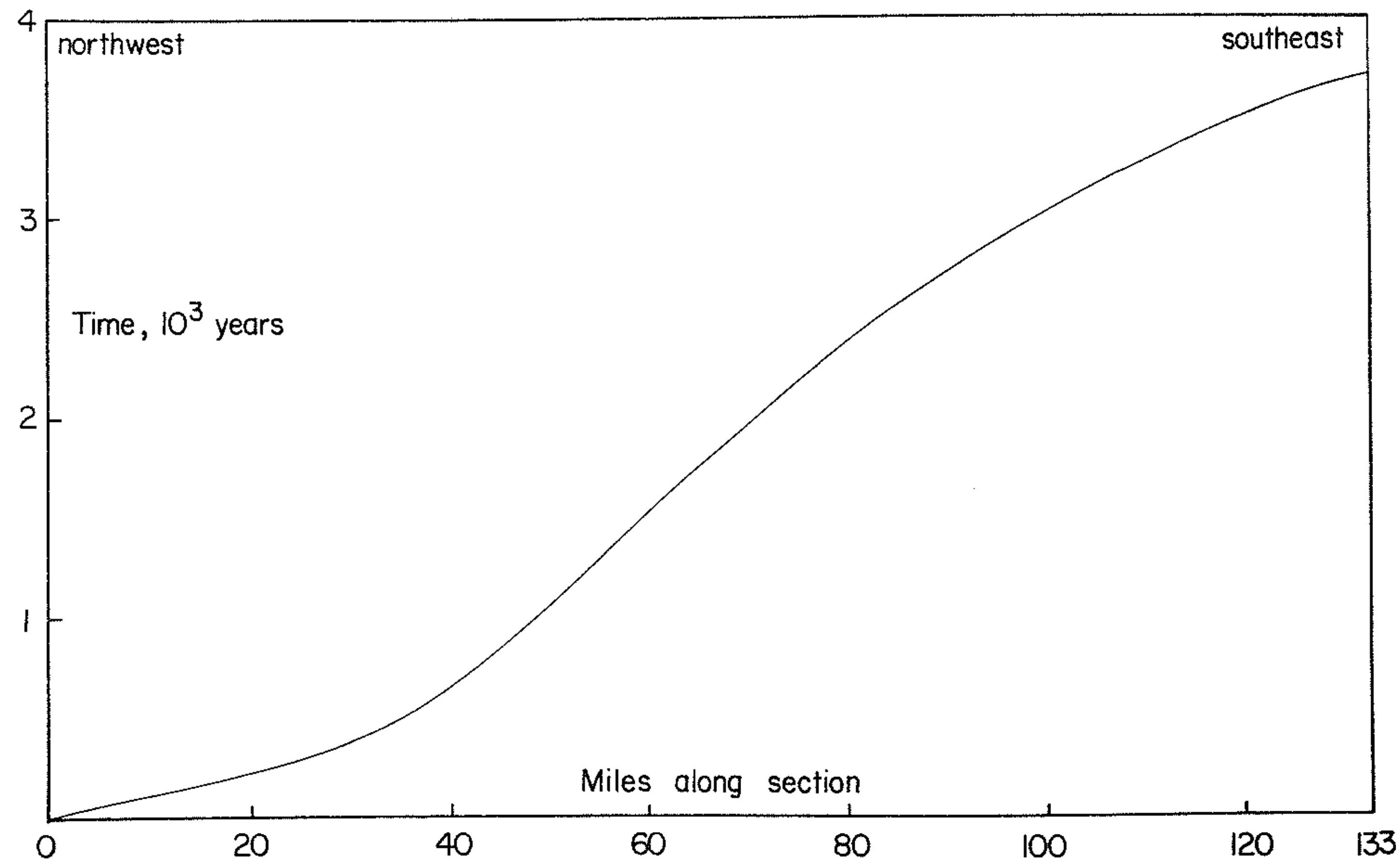


Figure 10: Residence time of average water particle in traversing the composite diagonal section of Figure 1.

composite diagonal section (Fig. 1). The speed of the particle is large at first, owing to the relatively high slope of the Ogallala base and of the water table on the New Mexico side of the section. In the middle of the section the velocity is minimal, owing to the flatness of the bottom as well as of the water table. Towards the eastern margin of the section the speed again increases, reaching its maximum value as the water emerges at the eastern scarp. Calculating this curve it was assumed that the porosity is 30 percent and that the hydraulic conductivity is 400 gpd/ft$^2$ or 19,500 ft$^2$/yr.

Exhaustion of Groundwater Reservoir Assuming Cessation of Recharge

In answer to Question 1, refer to Figures 11 and 12. Fig. 11 is a graph showing the decay of the water table profile along the idealized strip, assuming that recharge were to cease immediately. The value of the relative slope is taken to be 2.00.*) The curve marked $\tau = 0$ gives the initial profile, approximately as it was in 1938. Successive curves taken in steps of $\Delta\tau = 0.02$ are spaced roughly 200 years apart. Note that, as the water table declines, the upstream toe of the "nappe" of the aquifer migrates downslope at a more or less uniform rate.

Figure 12 shows the rate of outflow and the loss of stored water after cessation of recharge. These curves were obtained by solving numerical difference equations corresponding to the differential equation governing the groundwater flow in such systems (Eqs. 17 and 18).

*)The present writers question the validity of this assumption for the nonsteady-state case under discussion.

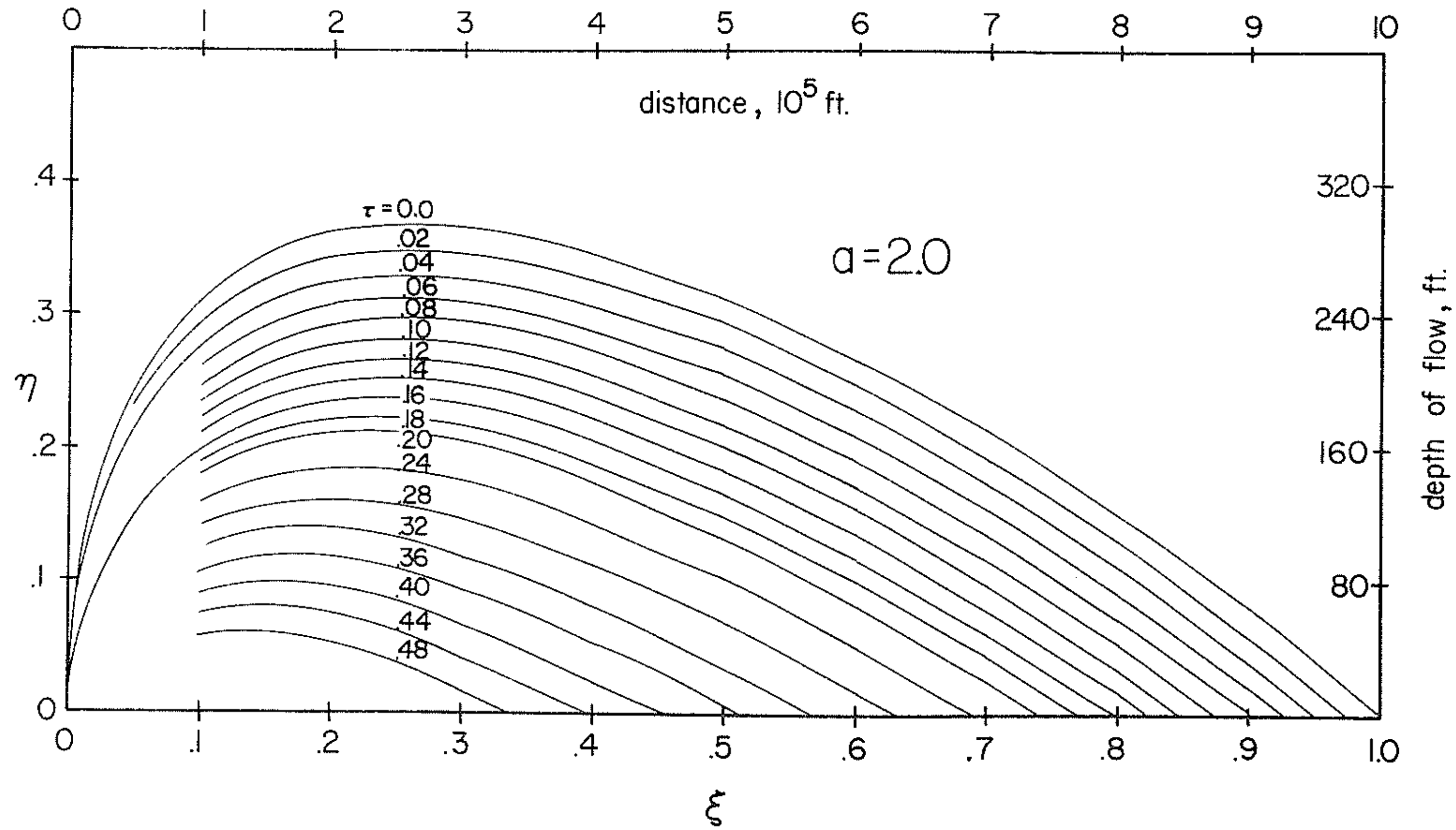


Figure 11: Decay of water table profile after cessation of recharge.

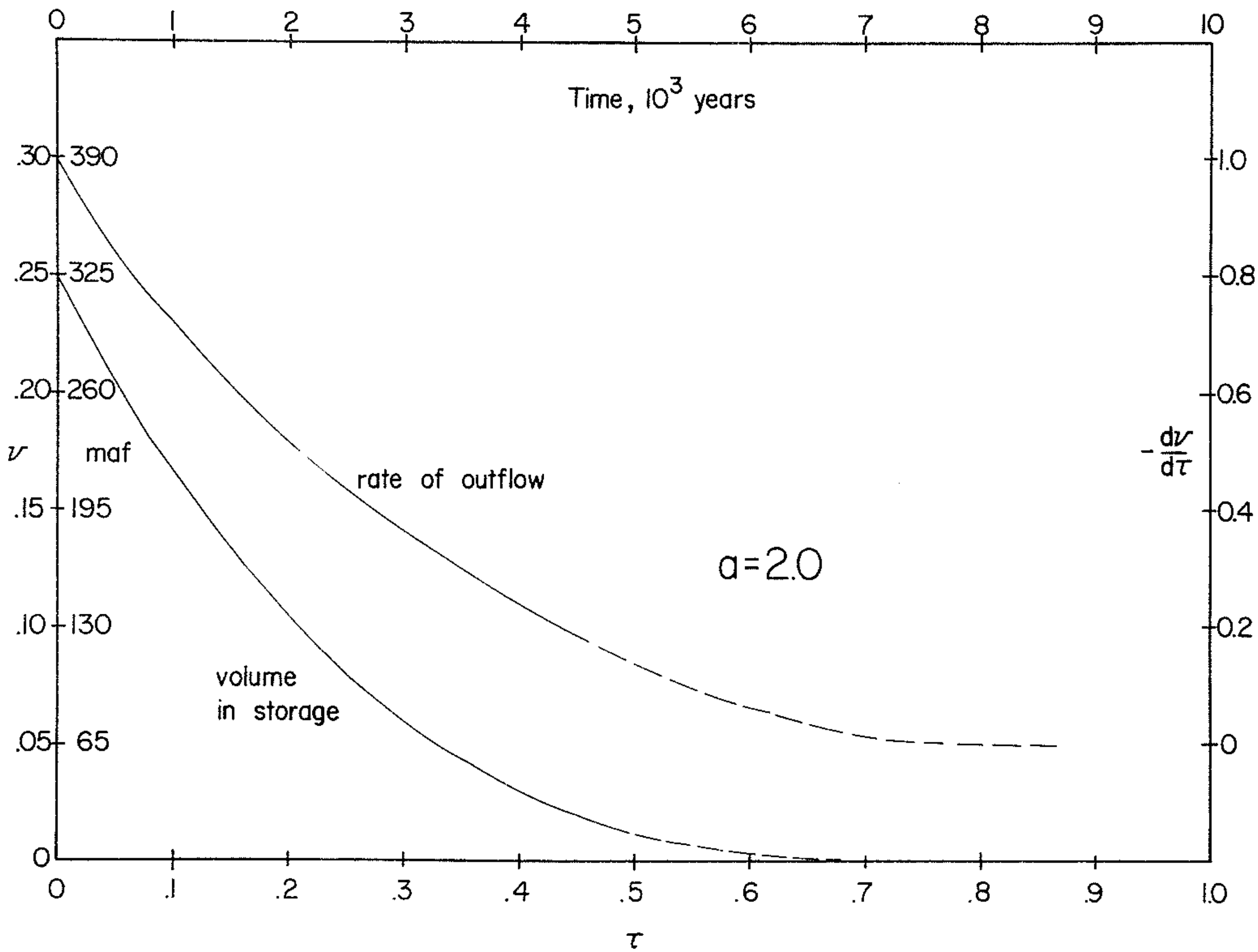


Figure 12: Rate of outflow and loss of storage after cessation of recharge.

It is concluded that the time required to deplete the reservoir to 5 percent of its original volume probably would not be in excess of 5,000 years. It is concluded that the time required to completely drain the idealized aquifer probably would not exceed 8,000 years.

Answering Question 1 specifically, that is as to "how long will it take the entire formation to reach the stage where no well will produce water", this requires further study in light of the unevenness of the bottom of the aquifer. It is possible that in certain areas where there are depressions in the buried Mesozoic landscape isolated wells might still be capable of producing water after the idealized strip aquifer has drained out.

Depletion of Groundwater at Recent Rates of Pumpage

Figure 13 is a graph showing the depletion of the groundwater storage under the Southern High Plains that would occur if pumping were to continue at various constant rates, presented in answer to Question 2. The rates chosen are respectively the minimum, average, and maximum annual rates during the period 1951-60. The minimum annual rate during this period was 2.10 maf, the average 4.45 maf, and the maximum 5.96 maf. We assumed that 15 percent of this gross pumpage was recirculated on the average, leaving as net pumpages the following figures: minimum, 1.78 maf per year; average, 3.78 maf per year; and maximum, 5.07 maf per year. Distributed over the 10.8 million acres of the selected area, these rates of withdrawal are equivalent respectively to 13.2, 28.0, and 37.6 times the assumed rate of natural recharge.

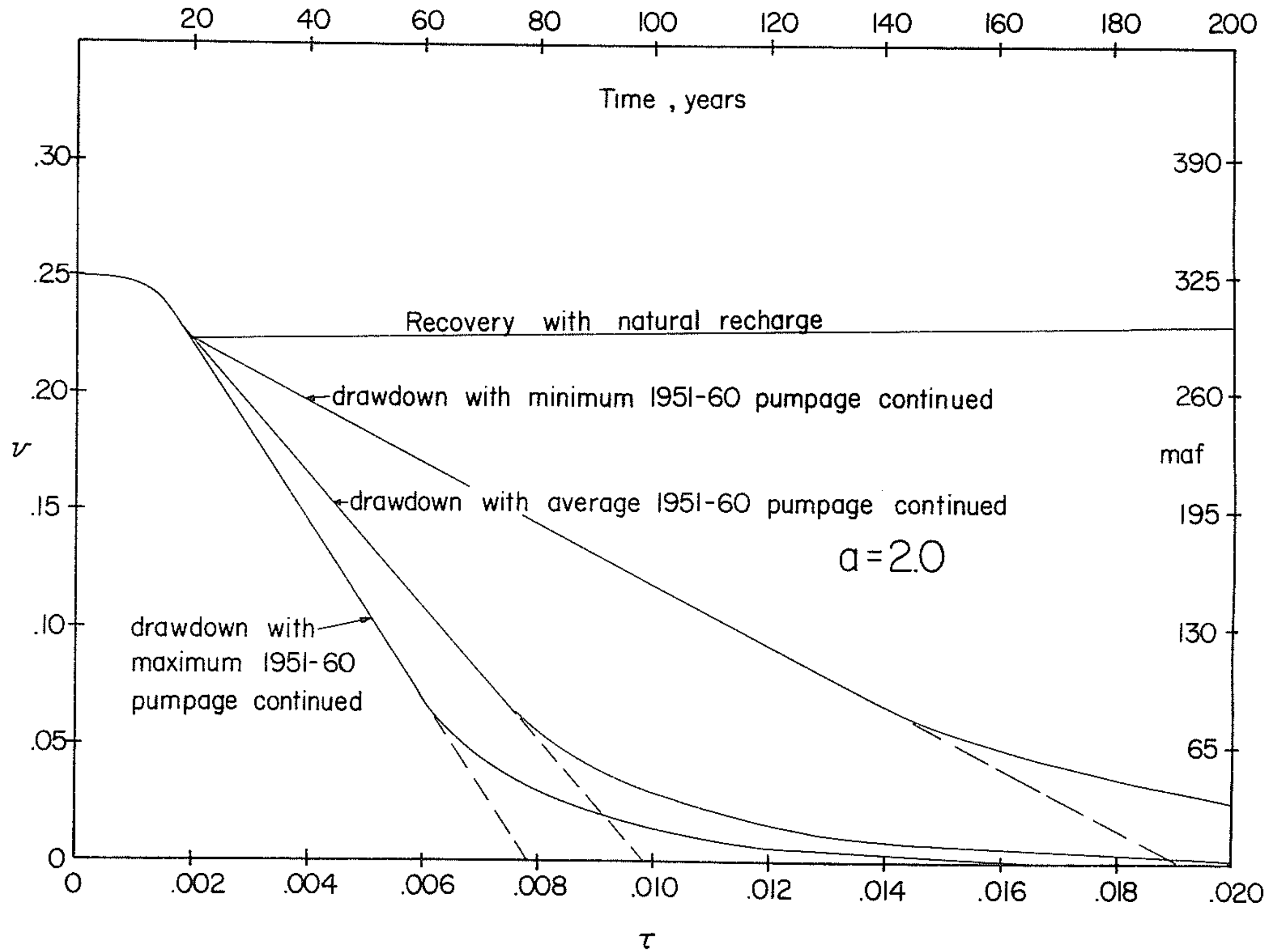


Figure 13: Depletion of groundwater storage at various constant rates of pumpage.

As Figure 13 based upon the idealized strip model, the depicted times are probably maximum times of drainage. In other words, if the pumpage were to continue at the average rate of 1951-60 it probably would not take longer than 100 years for the groundwater storage to be totally depleted, assuming that the density of wells were to be increased as needed near the end of the period of depletion. Similarly, with the maximum rate of pumpage for 1951-60 continued, it probably would not take more than about 80 years to completely exhaust the storage.

The curved lines near the bottom of Fig. 13 indicate the manner in which the storage would be depleted in each case if the present well density were to be maintained.

Restoration of Aquifer After Complete Exhaustion

In answer to Question 3 regarding the time required for the entire formation to return to its original saturated equilibrium conditions, refer to Figures 14 and 15. Fig. 14 shows successive water table profiles starting with the aquifer completely empty and building up the aquifer by uniform continuous recharge over a period of about 5,000 years (that is, to $\tau = 0.50$).

Figure 15 is a graph showing the restoration of storage after the postulated complete exhaustion. The flaring of the recovery curve toward its latter end is intended to suggest uncertainty in the extrapolation of the trend.

It is estimated that the time required to restore the groundwater

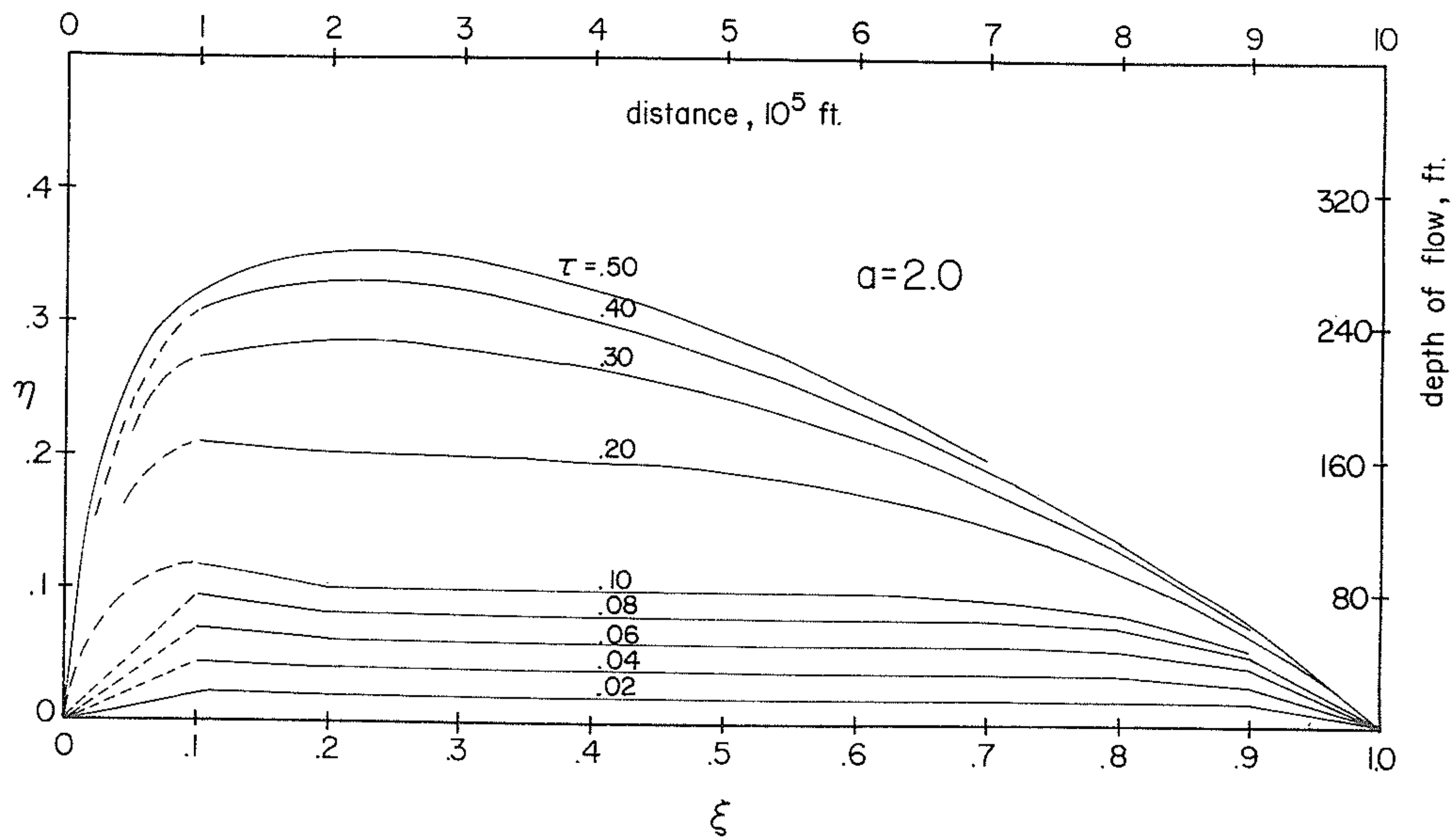


Figure 14: Restoration of water table profile after complete exhaustion.

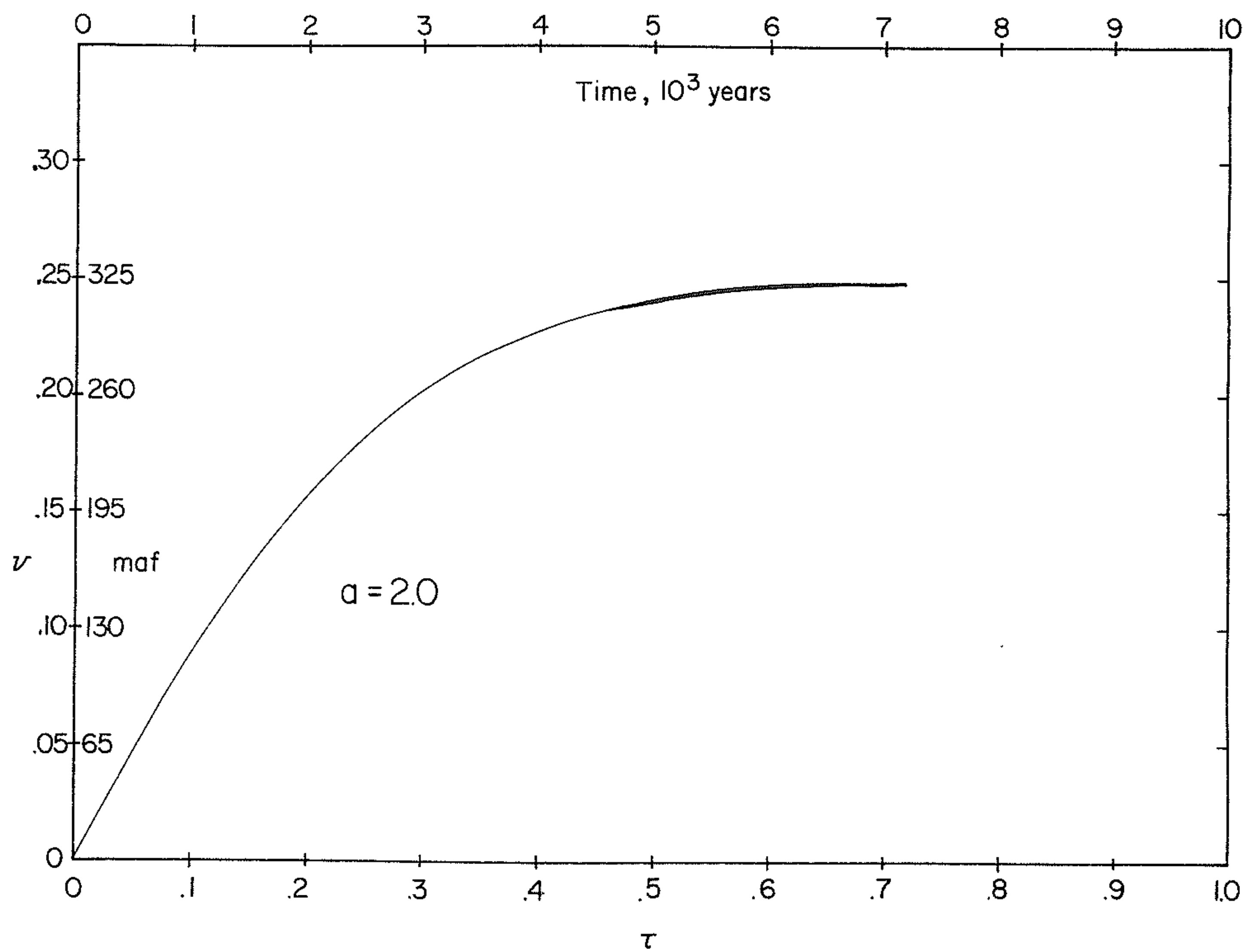


Figure 15: Restoration of storage after complete exhaustion.

reservoir to within 2 percent of its initial (1938) storage probably would not be in excess of something between 5,600 and 5,700 years. The time to restore it to within 1 percent of the initial storage probably would not exceed something between 6,200 and 6,500 years. Similarly, the time required to restore it to within ½ percent of its initial storage probably would not exceed something between 6,700 and 7,200 years.

Recovery of Partially Depleted Storage

Figure 16 shows the rate at which the present partially depleted storage would recover if pumping had ceased at the end of 1958. This is in answer to Question 4 as to "how long would it take to restore the formation to its original natural equilibrium", assuming no pumpage of any kind after the end of 1958 but assuming natural recharge.

As this graph was derived from the numerical solution of the difference equation for the idealized strip (Eq. 18), the times shown thereon are maximal. In other words, the time required for the aquifer to recover to within 2 percent of its initial stored volume probably would not exceed 1,500 years. Similarly, the time required for the aquifer to recover to within 1 percent of its initial stored volume probably would not exceed about 1,800 years; and to recover to within ½ percent, the time probably not to exceed 2,100 years

In further answer to Question 4 as to the depth of static water level under the Shurbet farm after full recovery, it may be said that theoretically it would take an infinite time for the formation to be

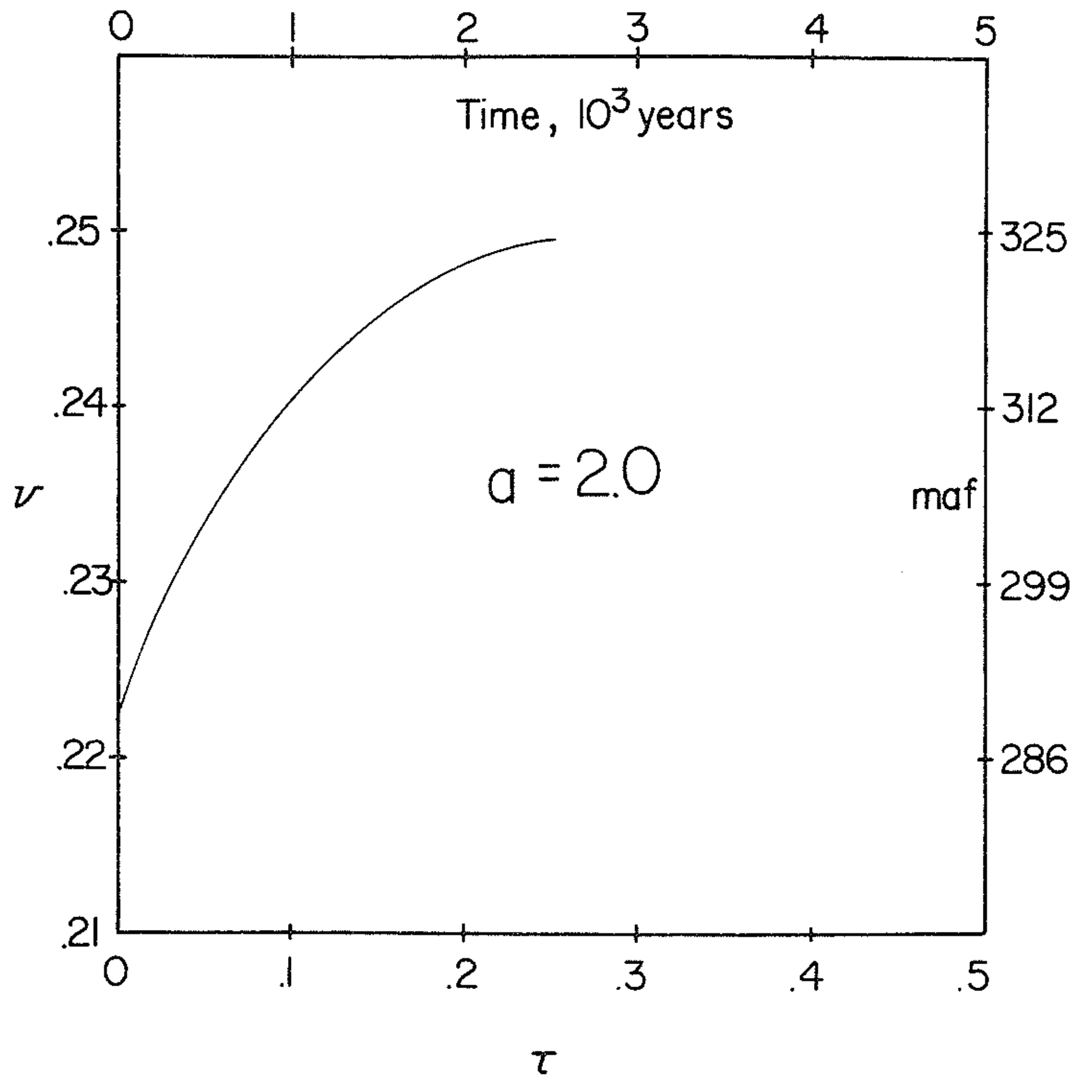


Figure 16: Recovery of depleted storage, assuming pumpage had ceased in 1958.

restored to its original natural equilibrium. At that time the depth of the static water level under the Shurbet farm would be the same as it was in 1938, setting aside any local or temporary climatic fluctuations of the water table.

Agreement Between Calculations and Observations

Because of the idealization of the uniformly sloping strip it is not to be expected that complete agreement can be obtained between the predictions based upon that strip and the observations already made and those that may be made in the future. However, by judicious selection of constants we have been able to obtain a fair agreement between the physical magnitudes involved.

The extent of the selected area of the Southern High Plains under study, $\underline{A}$, is $4.70 \times 10^{11}$ $ft^2$. The length of the idealized strip, $\underline{x_o}$, is $1.00 \times 10^6$ ft. Thus the average width of the problem at right angles to the strip is $\underline{A/x_o} = 4.70 \times 10^5$ ft. The area under the curve for $\underline{a} = 2.0$ (found in Figs. 4, 9, and 11) is given by the equation $\int xdy = 8.00 \times 10^8\ ft^2 \times \upsilon$. The area under the idealized profile for $\underline{a} = 2$ is 0.250 non-dimensional units. That is, $\upsilon^\circ = 0.250$. Thus, $(\int xdy)^\circ = 2.00 \times 10^8\ ft^2$.

To find the total volume in storage under the selected area of the Southern High Plains using the idealized strip as a guide, it is necessary to calculate the product $(A/x_o) \times (\int xdy)^\circ = 9.40 \times 10^{13}\ ft^3 = 2.16 \times 10^9$ af.

If every section drawn parallel to the selected profile were similar to that profile, and if the average specific yield is assumed to be 15 percent, the volume of drainable water may be calculated as follows: $(SA/x_o) \times (\int xdy)^{\circ} = 3.24 \times 10^8$ af. For convenience we designate this drainable volume by another symbol: $V^{\circ} = 324$ maf.

According to estimates by Mr. Broadhurst, the total volume of water pumped from the Southern High Plains between 1939 and 1958 was 42.8 maf. Assuming that 15 percent of this was recirculated on the average, the net pumpage during this 20-year period was 36.4 maf. This represents 11.2 percent of the equivalent drainable volume in storage.

Multiplying $v^{\circ}$ by this percentage, that is $0.250 \times 0.112$, we get 0.0280 for the reduction of $v$ corresponding to the 11.2 percent reduction in the apparent drainable volume under the Southern High Plains. By comparison, through the procedure of numerical analysis we get 0.0275 (see Fig. 9). The observed decline shown by the spread between the segmented dashed lines in Fig. 9 is on the other hand equivalent to 0.0271. The agreement here is good as far as the reduction of depth of flow or reduction of stored volume is concerned. However, as the actual volume in storage must be somewhat less than 324 maf (estimates range from 160 to 250 maf, depending upon the area selected), the actual volume reduction represents a greater fraction of the initial drainable volume in storage than the comparison on Fig. 9 would suggest. For this reason, as for the reasons noted above, the time decay constant of the aquifer probably is less than given in the foregoing curves (Figs 11 through 15).